

## Basic Notation

### Basic set notation

$\{a_1, \dots, a_r\}$	the set with elements $a_1, \dots, a_r$ .
$a \in S$	$a$ is in the set $S$ .
$S = T$	the sets $S$ and $T$ are equal, <i>i.e.</i> , every element of $S$ is in $T$ and every element of $T$ is in $S$ .
$S \subseteq T$	the set $S$ is a subset of the set $T$ , <i>i.e.</i> , every element of $S$ is also an element of $T$ .
$\exists a \in S \mathcal{P}(a)$	there exists an $a$ in $S$ for which the property $\mathcal{P}$ holds.
$\forall x \in S \mathcal{P}(x)$	property $\mathcal{P}$ holds for every element in $S$ .
$\{a \in S \mid \mathcal{P}(a)\}$	the set of all $a$ in $S$ for which $\mathcal{P}$ holds (the set $S$ is sometimes omitted if it can be determined from context).
$A \cup B$	union of sets, $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$ .
$A \cap B$	intersection of sets, $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$ .
$A \times B$	Cartesian product of two sets, $A \times B = \{(a, b) \mid a \in A, b \in B\}$ .

### Some specific sets

$\mathbf{R}$	the set of real numbers.
$\mathbf{R}^n$	the set of real $n$ -vectors ( $n \times 1$ matrices).
$\mathbf{R}^{1 \times n}$	the set of real $n$ -row-vectors ( $1 \times n$ matrices).
$\mathbf{R}^{m \times n}$	the set of real $m \times n$ matrices.
$j$	can mean $\sqrt{-1}$ , in the company of electrical engineers.
$i$	can mean $\sqrt{-1}$ , for normal people; $i$ is the polite term in mixed company ( <i>i.e.</i> , when non-electrical engineers are present).
$\mathbf{C}, \mathbf{C}^n, \mathbf{C}^{m \times n}$	the set of complex numbers, complex $n$ -vectors, complex $m \times n$ matrices.
$\mathbf{Z}$	the set of integers: $\mathbf{Z} = \{\dots, -1, 0, 1, \dots\}$ .
$\mathbf{R}_+$	the nonnegative real numbers, <i>i.e.</i> , $\mathbf{R}_+ = \{x \in \mathbf{R} \mid x \geq 0\}$ .
$[a, b], (a, b), [a, b), (a, b)$	the real intervals $\{x \mid a \leq x \leq b\}$ , $\{x \mid a < x \leq b\}$ , $\{x \mid a \leq x < b\}$ , and $\{x \mid a < x < b\}$ , respectively.

## Vectors and matrices

We use square brackets [ and ] to construct matrices and vectors, with white space delineating the entries in a row, and a new line indicating a new row. For example [1 2] is a row vector in  $\mathbf{R}^{1 \times 2}$ , and  $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$  is matrix in  $\mathbf{R}^{2 \times 3}$ .  $[1 \ 2]^T$  denotes a column vector, *i.e.*, an element of  $\mathbf{R}^{2 \times 1}$ , which we abbreviate as  $\mathbf{R}^2$ .

We use curved brackets ( and ) surrounding lists of entries, delineated by commas, as an alternative method to construct (column) vectors. Thus, we have three ways to denote a column vector:

$$(1, 2) = [1 \ 2]^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Note that in our notation scheme (which is fairly standard), [1, 2, 3] and (1 2 3) aren't used. We also use square and curved brackets to construct block matrices and vectors. For example if  $x, y, z \in \mathbf{R}^n$ , we have

$$\begin{bmatrix} x & y & z \end{bmatrix} \in \mathbf{R}^{n \times 3},$$

a matrix with columns  $x, y$ , and  $z$ . We can construct a block vector as

$$(x, y, z) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbf{R}^{3n}.$$

## Functions

The notation  $f : A \rightarrow B$  means that  $f$  is a function on the set  $A$  into the set  $B$ . The notation  $b = f(a)$  means  $b$  is the value of the function  $f$  at the point  $a$ , where  $a \in A$  and  $b \in B$ . The set  $A$  is called the *domain* of the function  $f$ ; it can thought of as the set of legal parameter values that can be passed to the function  $f$ . The set  $B$  is called the *codomain* (or sometimes range) of the function  $f$ ; it can thought of as a set that contains all possible returned values of the function  $f$ .

There are several ways to think of a function. The formal definition is that  $f$  is a subset of  $A \times B$ , with the property that for every  $a \in A$ , there is exactly one  $b \in B$  such that  $(a, b) \in f$ . We denote this as  $b = f(a)$ .

Perhaps a better way to think of a function is as a *black box* or (software) *function* or *subroutine*. The domain is the set of all legal values (or data types or structures) that can be passed to  $f$ . The codomain of  $f$  gives the data type or data structure of the values returned by  $f$ .

Thus  $f(a)$  is *meaningless* if  $a \notin A$ . If  $a \in A$ , then  $b = f(a)$  is an element of  $B$ . Also note that the *function* is denoted  $f$ ; it is *wrong* to say 'the function  $f(a)$ ' (since  $f(a)$  is an element

of  $B$ , not a function). Having said that, we do sometimes use sloppy notation such as ‘the function  $f(t) = t^3$ ’. To say this more clearly you could say ‘the function  $f : \mathbf{R} \rightarrow \mathbf{R}$  defined by  $f(t) = t^3$  for  $t \in \mathbf{R}$ ’.

## Examples

- $-0.1 \in \mathbf{R}$ ,  $\sqrt{2} \in \mathbf{R}_+$ ,  $1 - 2j \in \mathbf{C}$  (with  $j = \sqrt{-1}$ ).
- The matrix

$$A = \begin{bmatrix} 0.3 & 6.1 & -0.12 \\ 7.2 & 0 & 0.01 \end{bmatrix}$$

is an element in  $\mathbf{R}^{2 \times 3}$ . We can define a function  $f : \mathbf{R}^3 \rightarrow \mathbf{R}^2$  as  $f(x) = Ax$  for any  $x \in \mathbf{R}^3$ . If  $x \in \mathbf{R}^3$ , then  $f(x)$  is a particular vector in  $\mathbf{R}^2$ . We can say ‘the function  $f$  is linear’. To say ‘the function  $f(x)$  is linear’ is technically wrong since  $f(x)$  is a vector, not a function. Similarly we can’t say ‘ $A$  is linear’; it is just a matrix.

- We can define a function  $f : \{a \in \mathbf{R} \mid a \neq 0\} \times \mathbf{R}^n \rightarrow \mathbf{R}^n$  by  $f(a, x) = (1/a)x$ , for any  $a \in \mathbf{R}$ ,  $a \neq 0$ , and any  $x \in \mathbf{R}^n$ . The function  $f$  could be informally described as division of a vector by a nonzero scalar.
- Consider the set  $A = \{0, -1, 3.2\}$ . The elements of  $A$  are 0,  $-1$  and 3.2. Therefore, for example,  $-1 \in A$  and  $\{0, 3.2\} \subseteq A$ . Also, we can say that  $\forall x \in A$ ,  $-1 \leq x \leq 4$  or  $\exists x \in A$ ,  $x > 3$ .
- Suppose  $A = \{1, -1\}$ . Another representation for  $A$  is  $A = \{x \in \mathbf{R} \mid x^2 = 1\}$ .
- Suppose  $A = \{1, -2, 0\}$  and  $B = \{3, -2\}$ . Then

$$A \cup B = \{1, -2, 0, 3\}, \quad A \cap B = \{-2\}.$$

- Suppose  $A = \{1, -2, 0\}$  and  $B = \{1, 3\}$ . Then

$$A \times B = \{(1, 1), (1, 3), (-2, 1), (-2, 3), (0, 1), (0, 3)\}.$$

- $f : \mathbf{R} \rightarrow \mathbf{R}$  with  $f(x) = x^2 - x$  defines a function from  $\mathbf{R}$  to  $\mathbf{R}$  while  $u : \mathbf{R}_+ \rightarrow \mathbf{R}^2$  with

$$u(t) = \begin{bmatrix} t \cos t \\ 2e^{-t} \end{bmatrix}.$$

defines a function from  $\mathbf{R}_+$  to  $\mathbf{R}^2$ .