EE263 homework 7

1. *Some true/false questions.* Determine if the following statements are true or false. For each statement, either provide a proof that it is always true, or a counterexample demonstrating that it may fail.

   You can't assume anything about the dimensions of the matrices (unless it's explicitly stated), but you can assume that the dimensions are such that all expressions make sense. For example, the statement "$A + B = B + A$" is true, because no matter what the dimensions of A and B are (they must, however, be the same), and no matter what values A and B have, the statement holds. As another example, the statement $A^2 = A$ is false, because it fails for the matrix $\left[\begin{array}{c} 2 \end{array}\right]$. There are also matrices for which it does hold, *e.g.*, an identity matrix. But that doesn't make the statement true.

   (a) If $A \in \mathbf{R}^{3\times 3}$ satisfies $A + A^T = 0$, then A is singular.

   Solution. **True**. A general 3×3 skew symmetric matrix (*i.e.*, one that satisfies $A^T = -A$) has the form
$$A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}.$$
   Evidently we have $Ax = 0$, with $x = (c, -b, a)$. Alternatively, one can compute the determinant explicitly and show that it is identically zero.

   (b) If $A^k = 0$ for some integer $k \geq 1$, then $I - A$ is nonsingular.

   Solution. **True**. We first observe that all eigenvalues of A must be zero. The eigenvalues of $I - A$ are each one minus an eigevalue of A, *i.e.*, they are all equal to one. In particular, 0 is not an eigenvalue of $I - A$, so it is nonsingular.

   (c) If $A, B \in \mathbf{R}^{n\times n}$ are both diagonalizable, then AB is diagonalizable.

   Solution. **False**. Consider $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix}$. Clearly both A and B are diagonalizable, but $AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable.

   (d) If $A, B \in \mathbf{R}^{n\times n}$, then every eigenvalue of AB is an eigenvalue of BA.

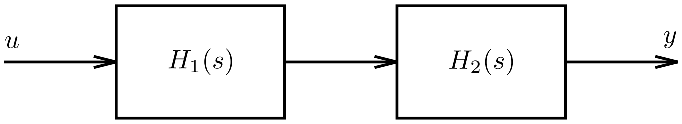
   Solution. **True**. Take any eigenvalue λ of AB, and let v be an eigenvector, *i.e.*, $ABv = \lambda v$. Suppose $\lambda \neq 0$, then
$$BA(Bv) = B(ABv) = \lambda(Bv).$$
   Since $Bv \neq 0$ (otherwise $\lambda = 0$), Bv is an eigenvector of BA associated with the eigenvalue λ. Now suppose $\lambda = 0$ and we need to show that BA is also singular. Suppose BA is nonsingular, then both A and B is full rank. But this will imply that $Bv = 0$ and thus $v = 0$, a contradiction.

   (e) If $A, B \in \mathbf{R}^{n\times n}$, then every eigenvector of AB is an eigenvector of BA.

Solution. **False.** Consider $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then $AB = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ has $(1, 1)$ as an eigenvector, which is clearly not an eigenvector of $BA = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$.

2. *Cascade connection of systems.*

   (a) Two linear systems (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) with states x_1 and x_2 (these are two *column vectors*, not two scalar components of one vector), have transfer functions $H_1(s)$ and $H_2(s)$, respectively. Find state equations for the cascade system:



   Use the state $x = \begin{bmatrix} x_1^T & x_2^T \end{bmatrix}^T$.

   (b) Use the state equations above to verify that the cascade system has transfer function $H_2(s)H_1(s)$. (To simplify, you can assume $D_1 = 0$, $D_2 = 0$.)

Solution.

(a) If y_1 is the output of the system with transfer function $H_1(s)$ (or input to the system with transfer function $H_2(s)$), for the first system we have

$$\dot{x}_1 = A_1 x_1 + B_1 u, \quad y_1 = C_1 x_1 + D_1 u$$

and for the second system

$$\begin{aligned} \dot{x}_2 = A_2 x_2 + B_2 y_1 = A_2 x_2 + B_2(C_1 x_1 + D_1 u) = B_2 C_1 x_1 + A_2 x_2 + B_2 D_1 u \\ y = C_2 x_2 + D_2 y_1 = C_2 x_2 + D_2(C_1 x_1 + D_1 u) = D_2 C_1 x_1 + C_2 x_2 + D_2 D_1 u. \end{aligned}$$

Therefore, the state equations for the cascade system become

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 D_1 \end{bmatrix} u \\ y &= \begin{bmatrix} D_2 C_1 & C_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + D_2 D_1 u \end{aligned}$$

(b) Using the state equations for the cascade system, the transfer function of the cascade system is

$$\begin{aligned} H_{\text{cascade}}(s) &= \begin{bmatrix} D_2 C_1 & C_2 \end{bmatrix}\left(sI - \begin{bmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{bmatrix}\right)^{-1}\begin{bmatrix} B_1 \\ B_2 D_1 \end{bmatrix} + D_2 D_1 \\ &= \begin{bmatrix} D_2 C_1 & C_2 \end{bmatrix}\begin{bmatrix} sI - A_1 & 0 \\ -B_2 C_1 & sI - A_2 \end{bmatrix}^{-1}\begin{bmatrix} B_1 \\ B_2 D_1 \end{bmatrix} + D_2 D_1. \end{aligned}$$

We will show that $H_{\text{cascade}}(s) = H_2(s)H_1(S)$. First, let us derive a formula for the inverse of a block-lower-triangular matrix. The goal is to find W, X, Y, Z such that:

$$\begin{bmatrix} P & 0 \\ Q & R \end{bmatrix}\begin{bmatrix} W & X \\ Y & Z \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

This leads to four equations that are easily solved:

$$\begin{array}{rcl} PW &=& I \quad \Longrightarrow \quad W = P^{-1} \\ PX &=& 0 \quad \Longrightarrow \quad X = 0 \\ QX + RZ &=& I \quad \Longrightarrow \quad Z = R^{-1} \\ QW + RY &=& 0 \quad \Longrightarrow \quad Y = -R^{-1}QP^{-1} \end{array}$$

So the inverse is simply:

$$\begin{bmatrix} W & X \\ Y & Z \end{bmatrix} = \begin{bmatrix} P & 0 \\ Q & R \end{bmatrix}^{-1} = \begin{bmatrix} P^{-1} & 0 \\ -R^{-1}QP^{-1} & R^{-1} \end{bmatrix}$$

Therefore

$$\begin{aligned} H_{\text{cascade}}(s) &= \begin{bmatrix} D_2C_1 & C_2 \end{bmatrix} \begin{bmatrix} (sI - A_1)^{-1} & 0 \\ (sI - A_2)^{-1}B_2C_1(sI - A_1)^{-1} & (sI - A_2)^{-1} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2D_1 \end{bmatrix} \\ &\quad + D_2D_1 \\ &= D_2C_1(sI - A_1)^{-1}B_1 + C_2(sI - A_2)^{-1}B_2C_1(sI - A_1)^{-1}B_1 \\ &\quad + C_2(sI - A_2)^{-1}B_2D_1 + D_2D_1 \\ &= D_2(H_1(s) - D_1) + (H_2(s) - D_2)(H_1(s) - D_1) + (H_2(s) - D_2)D_1 + D_2D_1 \\ &= H_2(s)H_1(s) \end{aligned}$$

and we are done.

3. *Periodic solution with intermittent input.* We consider the *stable* linear dynamical system $\dot{x} = Ax + Bu$, where $x(t) \in \mathbf{R}^n$, and $u(t) \in \mathbf{R}$. The input has the specific form

$$u(t) = \begin{cases} 1 & kT \le t < (k+\theta)T, \quad k = 0, 1, 2, \ldots \\ 0 & (k+\theta)T \le t < (k+1)T, \quad k = 0, 1, 2, \ldots \end{cases}$$

Here $T > 0$ is the *period*, and $\theta \in [0, 1]$ is called the *duty cycle* of the input. You can think of u as a constant input value one, that is applied over a fraction θ of each cycle, which lasts T seconds. Note that when $\theta = 0$, the input is $u(t) = 0$ for all t, and when $\theta = 1$, the input is $u(t) = 1$ for all t.

(a) Explain how to find an initial state $x(0)$ for which the resulting state trajectory is T-periodic, *i.e.*, $x(t+T) = x(t)$ for all $t \ge 0$. Give a formula for $x(0)$ in terms of the problem data, *i.e.*, A, B, T, and θ. Try to give the simplest possible formula.

(b) Explain why there is always *exactly one* value of $x(0)$ that results in $x(t)$ being T-periodic. In addition, explain why the formula you found in part (a) always makes sense and is valid. (For example, if your formula involves a matrix inverse, explain why the matrix to be inverted is nonsingular.)

Solution. For t between 0 and θT, we have $\dot{x} = Ax + B$. Therefore we have

$$\begin{aligned} x(\theta T) &= e^{\theta T A} x(0) + \left(\int_0^{\theta T} e^{tA} \, dt \right) B \\ &= e^{\theta T A} x(0) + A^{-1} \left(e^{\theta T A} - I \right) B \\ &= e^{\theta T A} (x(0) + A^{-1} B) - A^{-1} B, \end{aligned}$$

where we use, for example, the fact that A^{-1} and $e^{\theta T A}$ commute. For t between θT and T we have $\dot{x} = Ax$, so

$$\begin{aligned} x(T) &= e^{(T - \theta T)A} x(\theta T) \\ &= e^{TA} (x(0) + A^{-1} B) - e^{(1-\theta)TA} A^{-1} B. \end{aligned}$$

To have $x(0) = x(T)$ (which is all that's needed to have a periodic state trajectory), we must have

$$x(0) = e^{TA} (x(0) + A^{-1} B) - e^{(1-\theta)TA} A^{-1} B,$$

so

$$(I - e^{TA}) x(0) = e^{TA} A^{-1} B - e^{(1-\theta)TA} A^{-1} B = (I - e^{-\theta TA}) e^{TA} A^{-1} B.$$

Therefore we have

$$x(0) = (I - e^{TA})^{-1} (I - e^{-\theta TA}) e^{TA} A^{-1} B.$$

There are many other ways to write down this formula. In fact, any of the terms involving A can be freely moved around, since they all commute with each other. For example, another valid formula is

$$x(0) = A^{-1} (I - e^{-\theta TA}) (I - e^{TA})^{-1} e^{TA} B.$$

The inverse A^{-1} exists, since A is stable. The inverse $(I - e^{TA})^{-1}$ exists, for the following reason. The eigenvalues of $I - e^{TA}$ are $1 - e^{T\lambda_i}$, where λ_i are the eigenvalues of A. Since A is stable, these have negative real part, so we cannot have $1 - e^{T\lambda_i} = 0$. Thus, the matrix $I - e^{TA}$ is invertible. Here's an argument that's almost right, but not quite. Some people argued that $I - e^{TA}$ can only be singular if A has a zero eigenvalue. That's not correct; A can have an eigenvalue $2\pi j$, which is not zero, and that will cause $I - e^{TA}$ to be singular. (But note that because A is stable, it can't have a pure imaginary eigenvalue.)

(c) We now consider the specific system with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 8 \\ 2 \\ -14 \end{bmatrix}, \qquad T = 5.$$

Plot J, the mean-square norm of the state,

$$J = \frac{1}{T} \int_0^T \|x(t)\|^2 \, dt,$$

versus θ, for $0 \leq \theta \leq 1$, where $x(0)$ is the periodic initial condition that you found in part (a). You may approximate J as

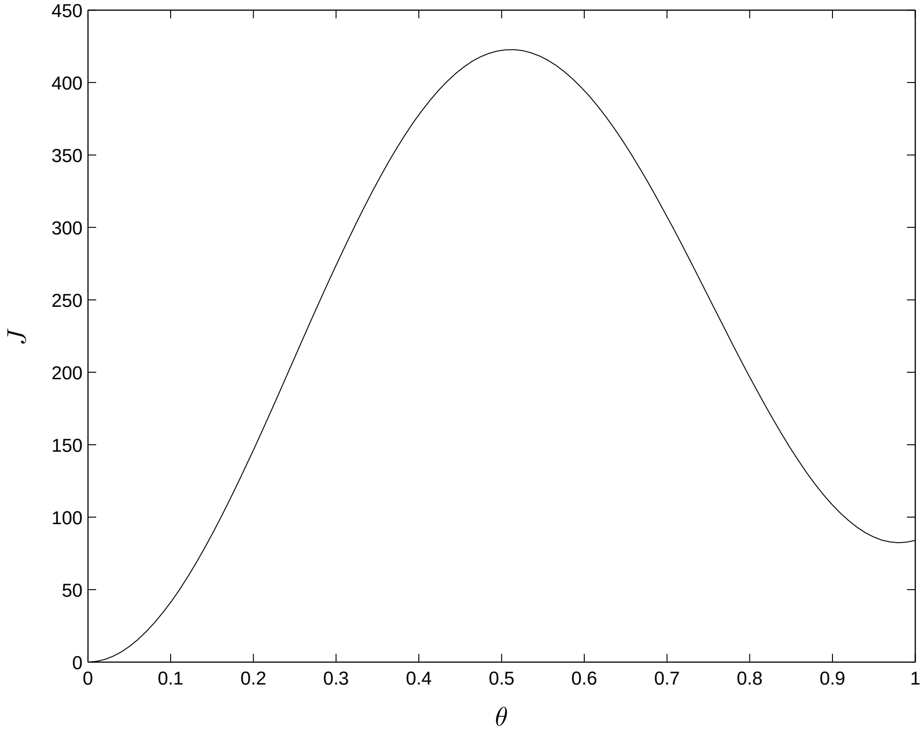
$$J \approx \frac{1}{T} \sum_{i=0}^{N-1} \|x(iT/N)\|^2,$$

for N large enough (say 1000). Estimate the value of θ that maximizes J.

Solution. The following Matlab code evaluates J for the given system for 100 values of θ equally spaced between 0 and 1:

```
clear
A = [0 1 0; 0 0 1; -1 -2 -1];
B = [8; 2; -14];
T = 5;
J = [];
for theta = linspace(0,1,100)
x_0 = inv(eye(3)-expm(T*A))*(eye(3)-expm(-theta*T*A))*expm(T*A)*inv(A)*B;
x_tT = expm(theta*T*A)*(x_0+inv(A)*B)-inv(A)*B;
Sx = 0;
for t = linspace(0,999*T/1000,1000)
if t < theta*T
x = expm(t*A)*(x_0+inv(A)*B)-inv(A)*B;
else
x = expm((t-theta*T)*A)*x_tT;
end
Sx = Sx+norm(x)^2;
end
J = [J Sx/1000];
end
plot(linspace(0,1,100),J)
xlabel('\theta')
ylabel('J')
print resonance.eps
```

This codes generates the following figure:



Evidently, J is maximized for $\theta \approx 0.5$.

4. *Positive semidefinite (PSD) matrices.*

   (a) Show that if A and B are PSD and $\alpha \in \mathbf{R}$, $\alpha \geq 0$, then so are αA and $A + B$.

   (b) Show that any (symmetric) submatrix of a PSD matrix is PSD. (To form a symmetric submatrix, choose any subset of $\{1, \ldots, n\}$ and then throw away all other columns and rows.)

   (c) Show that if $A \geq 0$, $A_{ii} \geq 0$.

   (d) Show that if $A \geq 0$, $|A_{ij}| \leq \sqrt{A_{ii}A_{jj}}$. In particular, if $A_{ii} = 0$, then the entire ith row and column of A are zero.

   Solution:

   (a) To show that $\alpha A \geq 0$ we verify that $x^T(\alpha A)x \geq 0$ for all x. But $x^T(\alpha A)x = \alpha(x^TAx)$ and since $x^TAx \geq 0$ ($A \geq 0$) and $\alpha \geq 0$, we immediately get $x^T(\alpha A)x \geq 0$. Again, to show that $A + B \geq 0$ we show that $x^T(A+B)x \geq 0$ for all x. This is easy because $x^T(A+B)x = x^TAx + x^TBx$ and $A, B \geq 0$ imply that $x^TAx, x^TBx \geq 0$ and therefore $x^T(A+B)x \geq 0$.

   (b) Suppose that $A = A^T \geq 0$. Any symmetric submatrix of A can be written as Z^TAZ for some suitable matrix Z. For example, if $A \in \mathbf{R}^{3\times 3}$ and we want to pick the submatrix formed by the first and third columns and rows we simply take

   $$Z = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

   so that

   $$Z^TAZ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{13} \\ A_{13} & A_{33} \end{bmatrix}.$$

The idea here is to pick the columns of Z as the unit vectors corresponding to the column/row numbers we want to keep. In this example, we wanted to keep the first and third columns/rows so we took $Z = [e_1\ e_3]$. In general, consider the $m \times m$ symmetric submatrix of A which consists of elements of A that are only on the columns and rows $i_1, \ldots, i_m$ of A. Then it is easy to verify that

$$(\text{submatrix formed from columns/rows } i_1, \ldots, i_m) = Z^T A Z, \quad Z = [e_{i_1} \ \cdots \ e_{i_m}],$$

where e_{i_j} is the i_jth unit vector in $\mathbf{R}^n$. Using the result of problem (**??**), $A \geq 0$ implies that $Z^T A Z \geq 0$ and therefore any symmetric submatrix of A is also positive semidefinite.

(c) This is easy. We can simply use the result of the previous part ($A_{ii} \in \mathbf{R}$ is a 1×1 symmetric submatrix of A), or more directly, use the fact that $A \geq 0$ implies $e_i^T A e_i \geq 0$ and note that $e_i^T A e_i$ is nothing but A_{ii}.

(d) Choose any 2×2 symmetric submatrix of A, say

$$\tilde{A} = \begin{bmatrix} A_{ii} & A_{ij} \\ A_{ij} & A_{jj} \end{bmatrix}$$

According to problem (4b) this (symmetric) submatrix is positive semidefinite and therefore its eigenvalues are nonnegative. Hence, the determinant of the submatrix (which is equal to the product of the eigenvalues) is also nonnegative. In other words

$$\det \tilde{A} = \begin{bmatrix} A_{ii} & A_{ij} \\ A_{ij} & A_{jj} \end{bmatrix} = A_{ii}A_{jj} - A_{ij}^2 \geq 0$$

and immediately we get $A_{ij}^2 \leq A_{ii}A_{jj}$ or $|A_{ij}| \leq \sqrt{A_{ii}A_{jj}}$. In particular, if $A_{ii} = 0$ then $|A_{ij}| \leq 0$ or $A_{ij} = 0$ (for any j) and the entire ith row (and hence ith column since A is symmetric) should be zero.

5. Suppose A and B are symmetric matrices that yield the same quadratic form, *i.e.*, $x^T A x = x^T B x$ for all x. Show that $A = B$. *Hint:* first try $x = e_i$ (the ith unit vector) to conclude that the entries of A and B on the diagonal are the same; then try $x = e_i + e_j$.

Solution. With $x = e_i$, $x^T A x = x^T B x$ gives

$$A_{ii} = B_{ii}, \quad i = 1, 2, ..., n.$$

With $x = e_i + e_j$, $x^T A x = x^T B x$ gives

$$e_i^T A e_i + e_i^T A e_j + e_j^T A e_i + e_j^T A e_j = e_i^T B e_i + e_i^T B e_j + e_j^T B e_i + e_j^T B e_j$$

and therefore

$$A_{ii} + A_{ij} + A_{ji} + A_{jj} = B_{ii} + B_{ij} + B_{ji} + B_{jj}, \quad i, j = 1, 2, ..., n,$$

so that $A_{ij} + A_{ji} = B_{ij} + B_{ji}$. This, along with $A_{ii} = B_{ii}$ means that $A + A^T = B + B^T$. Finally, using the fact that $A = A^T$ and $B = B^T$, we conclude that $A = B$.