EE263 homework 7

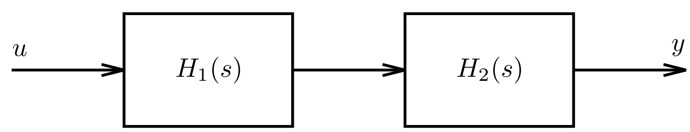
1. *Some true/false questions.* Determine if the following statements are true or false. For each statement, either provide a proof that it is always true, or a counterexample demonstrating that it may fail.

   You can't assume anything about the dimensions of the matrices (unless it's explicitly stated), but you can assume that the dimensions are such that all expressions make sense. For example, the statement "$A+B=B+A$" is true, because no matter what the dimensions of A and B are (they must, however, be the same), and no matter what values A and B have, the statement holds. As another example, the statement $A^2 = A$ is false, because it fails for the matrix $\left[\ 2\ \right]$. There are also matrices for which it does hold, *e.g.*, an identity matrix. But that doesn't make the statement true.

   (a) If $A \in \mathbf{R}^{3\times 3}$ satisfies $A + A^T = 0$, then A is singular.
   (b) If $A^k = 0$ for some integer $k \geq 1$, then $I - A$ is nonsingular.
   (c) If $A, B \in \mathbf{R}^{n\times n}$ are both diagonalizable, then AB is diagonalizable.
   (d) If $A, B \in \mathbf{R}^{n\times n}$, then every eigenvalue of AB is an eigenvalue of BA.
   (e) If $A, B \in \mathbf{R}^{n\times n}$, then every eigenvector of AB is an eigenvector of BA.

2. *Cascade connection of systems.*

   (a) Two linear systems (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) with states x_1 and x_2 (these are two *column vectors*, not two scalar components of one vector), have transfer functions $H_1(s)$ and $H_2(s)$, respectively. Find state equations for the cascade system:



   Use the state $x = \left[\ x_1^T \quad x_2^T\ \right]^T$.

   (b) Use the state equations above to verify that the cascade system has transfer function $H_2(s)H_1(s)$. (To simplify, you can assume $D_1 = 0$, $D_2 = 0$.)

3. *Periodic solution with intermittent input.* We consider the *stable* linear dynamical system $\dot{x} = Ax + Bu$, where $x(t) \in \mathbf{R}^n$, and $u(t) \in \mathbf{R}$. The input has the specific form

$$u(t) = \begin{cases} 1 & kT \leq t < (k+\theta)T, \quad k = 0, 1, 2, \ldots \\ 0 & (k+\theta)T \leq t < (k+1)T, \quad k = 0, 1, 2, \ldots \end{cases}$$

Here $T > 0$ is the *period*, and $\theta \in [0, 1]$ is called the *duty cycle* of the input. You can think of u as a constant input value one, that is applied over a fraction θ of each cycle, which lasts T seconds. Note that when $\theta = 0$, the input is $u(t) = 0$ for all t, and when $\theta = 1$, the input is $u(t) = 1$ for all t.

(a) Explain how to find an initial state $x(0)$ for which the resulting state trajectory is T-periodic, *i.e.*, $x(t+T) = x(t)$ for all $t \geq 0$. Give a formula for $x(0)$ in terms of the problem data, *i.e.*, A, B, T, and θ. Try to give the simplest possible formula.

(b) Explain why there is always *exactly one* value of $x(0)$ that results in $x(t)$ being T-periodic. In addition, explain why the formula you found in part (a) always makes sense and is valid. (For example, if your formula involves a matrix inverse, explain why the matrix to be inverted is nonsingular.)

(c) We now consider the specific system with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 8 \\ 2 \\ -14 \end{bmatrix}, \qquad T = 5.$$

Plot J, the mean-square norm of the state,

$$J = \frac{1}{T} \int_0^T \|x(t)\|^2 \, dt,$$

versus θ, for $0 \leq \theta \leq 1$, where $x(0)$ is the periodic initial condition that you found in part (a). You may approximate J as

$$J \approx \frac{1}{T} \sum_{i=0}^{N-1} \|x(iT/N)\|^2,$$

for N large enough (say 1000). Estimate the value of θ that maximizes J.

4. *Positive semidefinite (PSD) matrices.*

(a) Show that if A and B are PSD and $\alpha \in \mathbf{R}$, $\alpha \geq 0$, then so are αA and $A + B$.

(b) Show that any (symmetric) submatrix of a PSD matrix is PSD. (To form a symmetric submatrix, choose any subset of $\{1, \ldots, n\}$ and then throw away all other columns and rows.)

(c) Show that if $A \geq 0$, $A_{ii} \geq 0$.

(d) Show that if $A \geq 0$, $|A_{ij}| \leq \sqrt{A_{ii} A_{jj}}$. In particular, if $A_{ii} = 0$, then the entire ith row and column of A are zero.

5. Suppose A and B are symmetric matrices that yield the same quadratic form, *i.e.*, $x^T A x = x^T B x$ for all x. Show that $A = B$. *Hint:* first try $x = e_i$ (the ith unit vector) to conclude that the entries of A and B on the diagonal are the same; then try $x = e_i + e_j$.