EE263 homework 6

1. *Optimal choice of initial temperature profile.* We consider a thermal system described by an n-element finite-element model. The elements are arranged in a line, with the temperature of element i at time t denoted $T_i(t)$. Temperature is measured in degrees Celsius above ambient; negative $T_i(t)$ corresponds to a temperature below ambient. The dynamics of the system are described by

$$c_1\dot{T}_1 = -a_1 T_1 - b_1(T_1 - T_2),$$

$$c_i\dot{T}_i = -a_i T_i - b_i(T_i - T_{i+1}) - b_{i-1}(T_i - T_{i-1}), \quad i = 2, \ldots, n-1,$$

and

$$c_n\dot{T}_n = -a_n T_n - b_{n-1}(T_n - T_{n-1}).$$

where $c \in \mathbf{R}^n$, $a \in \mathbf{R}^n$, and $b \in \mathbf{R}^{n-1}$ are given and are all positive.

We can interpret this model as follows. The parameter c_i is the heat capacity of element i, so $c_i\dot{T}_i$ is the net heat flow into element i. The parameter a_i gives the thermal conductance between element i and the environment, so $a_i T_i$ is the heat flow from element i to the environment (*i.e.*, the direct heat loss from element i.) The parameter b_i gives the thermal conductance between element i and element $i+1$, so $b_i(T_i - T_{i+1})$ is the heat flow from element i to element $i+1$. Finally, $b_{i-1}(T_i - T_{i-1})$ is the heat flow from element i to element $i-1$.

The goal of this problem is to choose the initial temperature profile, $T(0) \in \mathbf{R}^n$, so that $T(t^{\text{des}}) \approx T^{\text{des}}$. Here, $t^{\text{des}} \in \mathbf{R}$ is a specific time when we want the temperature profile to closely match $T^{\text{des}} \in \mathbf{R}^n$. We also wish to satisfy a constraint that $\|T(0)\|$ should be not be too large.

To formalize these requirements, we use the objective $(1/\sqrt{n})\|T(t^{\text{des}}) - T^{\text{des}}\|$ and the constraint $(1/\sqrt{n})\|T(0)\| \leq T^{\max}$. The first expression is the RMS temperature deviation, at $t = t^{\text{des}}$, from the desired value, and the second is the RMS temperature deviation from ambient at $t = 0$. $T^{\max}$ is the (given) maximum inital RMS temperature value.

(a) Explain how to find $T(0)$ that minimizes the objective while satisfying the constraint.

(b) Solve the problem instance with the values of n, c, a, b, t_{des}, T^{des} and $T^{\max}$ defined in the file `temp_prof_data.m`.
Plot, on one graph, your $T(0)$, $T(t^{\text{des}})$ and T^{des}. Give the RMS temperature error $(1/\sqrt{n})\|T(t^{\text{des}}) - T^{\text{des}}\|$, and the RMS value of initial temperature $(1/\sqrt{n})\|T(0)\|$.

Solution.

(a) We can express the temperature dynamics as $\dot{T} = AT$, where A is a tridiagonal matrix with

$$\begin{aligned} A_{11} &= -1/c_1(a_1 + b_1) \\ A_{ii} &= -1/c_i(a_i + b_i + b_{i-1}), \quad i = 2, \ldots, n, \\ A_{i,i-1} &= b_{i-1}/c_i, \quad i = 2, \ldots, n, \\ A_{i,i+1} &= b_i/c_i, \quad i = 1, \ldots, n-1. \end{aligned}$$

We have $T(t^{\text{des}}) = e^{t^{\text{des}}A}T(0)$. Therefore we must solve the problem

$$\begin{array}{ll} \text{minimize} & (1/n)\left\| e^{t^{\text{des}}A}T(0) - T^{\text{des}} \right\|^2 \\ \text{subject to} & (1/n)\|T(0)\|^2 \leq (T^{\max})^2. \end{array}$$

We solve this by minimizing

$$\left\| e^{t^{\mathrm{des}}A} T(0) - T^{\mathrm{des}} \right\|^2 + \rho \|T(0)\|^2,$$

and increasing ρ until $(1/n)\|T(0)\|^2 \leq (T^{\max})^2$ first holds (which will be with equality).

We mention one rather common error: simply obtaining the least-squares solution (without regards for $\|T(0)\|$, and then scaling this solution down so that the constraint is satisfied. This method produces results that look pretty good, when plotted, but are in fact not particularly good (in addition to just being wrong). This results in an RMS temperature error that more than 60% higher than using the correct method.

(b) The following code solves the problem in part (b).

```
clear all;  close all

temp_prof_data

A(1,1)=-1/c(1)*(a(1)+b(1));
A(1,2)=b(1)/c(1);
A(n,n)=-1/c(n)*(a(n)+b(n-1));
A(n,n-1)=b(n-1)/c(n);
for i=2:n-1
    A(i,i)=-1/c(i)*(a(i)+b(i)+b(i-1));
    A(i,i-1)=b(i-1)/c(i);
    A(i,i+1)=b(i)/c(i);
end


B=expm(A*tdes);
rho=0;
while 1
    C=[B; sqrt(rho)*eye(n)];
    d=[Tdes;zeros(n,1)];
    T=C\d;
    if norm(T)/sqrt(n)<=Tmax
        break
    end
    rho=rho+1e-5;
end

plot(Tdes); hold on
plot(B*T,'xr');
plot(T,'k'); hold off
set(gca,'Fontname','Times','Fontsize', 16);
ylabel('t'); xlabel('x')
print -depsc temp_prof

norm(T)/sqrt(n)
norm(Tdes-B*T)/sqrt(n)
```

Figure **??** shows T^{des} with a solid blue line, $T(t^{\mathrm{des}})$ with a dashed red line and $T(0)$ with a dash-dotted black line. We have $(1/\sqrt{n})\|T(t^{\mathrm{des}}) - T^{\mathrm{des}}\| = 0.0457$, and, as expected, $(1/\sqrt{n})\|T(0)\| = 2.50$.

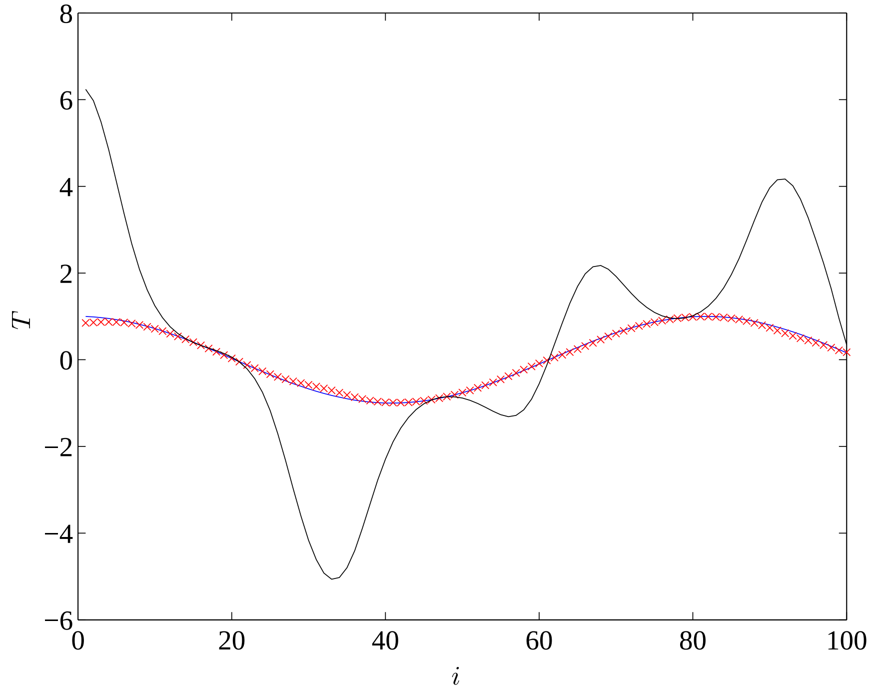

Figure 1: $T^{\rm des}$ (solid blue), $T(t^{\rm des})$ (dashed red) and $T(0)$ (dash-dotted black).

2. *Positive quadrant invariance.* We consider a system $\dot{x} = Ax$ with $x(t) \in \mathbf{R}^2$ (although the results of this problem can be generalized to systems of higher dimension). We say the system is *positive quadrant invariant* (PQI) if whenever $x_1(T) \geq 0$ and $x_2(T) \geq 0$, we have $x_1(t) \geq 0$ and $x_2(t) \geq 0$ for all $t \geq T$. In other words, if the state starts inside (or enters) the positive (*i.e.*, first) quadrant, then the state remains indefinitely in the positive quadrant.

   (a) Find the precise conditions on A under which the system $\dot{x} = Ax$ is PQI. Try to express the conditions in the simplest form.

   (b) *True or False:* if $\dot{x} = Ax$ is PQI, then the eigenvalues of A are real.

Solution.

(a) This problem can be solved by several methods. The simplest method is probably this: at the quadrant boundaries, the derivative $\dot{x} = Ax$ must point *into* the quadrant (or at least, along the quadrant boundary). Therefore, if n is the *inward* normal vector to the boundary at coordinates x we must have $n^T Ax \geq 0$. The first boundary is characterized by $x_1 = 0$ and $x_2 \geq 0$. The inward normal vector at all points of this boundary is simply $n = e_1 = [1\ 0]^T$. Therefore we should have $e_1^T Ax \geq 0$. Similarly, for the other boundary characterized by $x_1 \geq 0$ and $x_2 = 0$ we require $e_2^T Ax \geq 0$ If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then the conditions become

$$\begin{array}{l} x_1 = 0,\ x_2 \geq 0,\ e_1^T Ax = a_{11}x_1 + a_{12}x_2 = a_{12}x_2 \geq 0 \implies a_{12} \geq 0, \\ x_2 = 0,\ x_1 \geq 0,\ e_2^T Ax = a_{21}x_2 + a_{22}x_2 = a_{21}x_1 \geq 0 \implies a_{21} \geq 0. \end{array}$$

In other words, the off-diagonal elements must be non-negative. In fact, systems $\dot{x} = Ax$ of *any* degree are *positive orthant invariant* (POI) if and only if all off-diagonal elements of A are nonnegative. The problem can also be solved by studying whether all entries of the matrix e^{At} are nonnegative. That is quite a bit more complicated, but ends with the same condition.

(b) *True.* The characteristic equation of the system is

$$\det(sI - A) = \det \begin{bmatrix} s - a_{11} & -a_{12} \\ -a_{21} & s - a_{22} \end{bmatrix} = s^2 - (a_{11} + a_{22})s + a_{11}a_{22} - a_{12}a_{21}.$$

In order for the eigenvalues to be complex, the discriminant of the quadratic equation, $b^2 - 4ac$, must be negative. But

$$(a_{11} + a_{22})^2 - 4a_{11}a_{22} + 4a_{12}a_{21} = (a_{11} - a_{22})^2 + 4a_{12}a_{21} \geq 0.$$

So, the eigenvalues must indeed be real. We can also derive this result another way. If the system has complex eigenvalues, then x_1 has the form $x_1(t) = ae^{\sigma t}\cos(\omega t + \phi)$ (and similarly for x_2). Obviously then there are positive times when $x_1(t) < 0$, contradicting PQI.

3. *Properties of the matrix exponential.*

(a) Show that $e^{A+B} = e^A e^B$ if A and B commute, *i.e.*, $AB = BA$.

(b) Carefully show that $\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$.

Solution.

(a) We will show that if A and B commute then $e^A e^B = e^{A+B}$. We begin by writing the expressions for e^A and e^B

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$
$$e^B = I + B + \frac{B^2}{2!} + \frac{B^3}{3!} + \cdots$$

Now we multiply both expressions and get

$$\begin{aligned} e^A e^B &= I + A + B + AB + \frac{A^2}{2!} + \frac{B^2}{2!} + \frac{A^3}{3!} + \frac{A^2 B}{2!} + \frac{AB^2}{2!} + \frac{B^3}{3!} + \cdots \\ &= I + A + B + \frac{A^2 + 2AB + B^2}{2!} + \frac{A^3 + 3A^2B + 3AB^2 + B^3}{3!} + \cdots \end{aligned}$$

Now we note that when A and B commute, we are able to write things such as $(A+B)^2 = A^2 + 2AB + B^2$. Otherwise, we would have: $(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2$. So, if A and B commute we can finally write

$$e^A e^B = I + (A+B) + \frac{(A+B)^2}{2!} + \frac{(A+B)^3}{3!} + \cdots = e^{A+B}$$

Alternate solution: We can be more rigorous by using summation notation:

$$\begin{aligned}
e^{A+B} &= \sum_{n=0}^{\infty} \frac{1}{n!}(A+B)^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} A^{n-k} B^k && \text{[binomial thm holds since } A \text{ and } B \text{ commute]} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{(n-k)!} \frac{1}{k!} A^{n-k} B^k \\
&= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{1}{(n-k)!} \frac{1}{k!} A^{n-k} B^k && \text{[reorder the two summations]} \\
&= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{m!} \frac{1}{k!} A^m B^k && \text{[let } m = n-k\text{]} \\
&= \left(\sum_{m=0}^{\infty} \frac{1}{m!} A^m \right) \left(\sum_{k=0}^{\infty} \frac{1}{k!} B^k \right) \\
&= e^A e^B
\end{aligned}$$

(b) It suffices to note that A commute with itself. Then one can write

$$\begin{aligned}
\frac{de^{At}}{dt} &= A + A^2 t + \frac{A^3 t^2}{2!} + \cdots \\
&= A(I + At + \frac{(At)^2}{2!} + \cdots) \\
&= (I + At + \frac{(At)^2}{2!} + \cdots)A \\
&= Ae^{At} = e^{At}A
\end{aligned}$$

4. *A simple population model.* We consider a certain population of fish (say) each (yearly) season. $x(t) \in \mathbf{R}^3$ will describe the population of fish at year $t \in \mathbf{Z}$, as follows:

- $x_1(t)$ denotes the number of fish less than one year old
- $x_2(t)$ denotes the number of fish between one and two years old
- $x_3(t)$ denotes the number of fish between two and three years

(We will ignore the fact that these numbers are integers.) The population evolves from year t to year $t+1$ as follows.

- The number of fish less than one year old in the next year $(t+1)$ is equal to the total number of offspring born during the current year. Fish that are less than one year old in the current year (t) bear no offspring. Fish that are between one and two years old in the current year (t) bear an average of 2 offspring each. Fish that are between two and three years old in the current year (t) bear an average of 1 offspring each.
- 40% of the fish less than one year old in the current year (t) die; the remaining 60% live on to be between one and two years old in the next year $(t+1)$.
- 30% of the one-to-two year old fish in the current year die, and 70% live on to be two-to-three year old fish in the next year.
- All of the two-to-three year old fish in the current year die.

Express the population dynamics as a discrete autonomous linear system with state $x(t)$, *i.e.*, in the form $x(t+1) = Ax(t)$. **Remark:** this example is silly, but more sophisticated population dynamics models are very useful and widely used.

Solution. We need to describe the population of fish at year $t+1$, *i.e.*, $x_1(t+1)$, $x_2(t+1)$ and $x_3(t+1)$ in terms of the population of fish at year t.

- The number of young fish at year $t+1$ is a result from offspring from both older populations of fish at year t. Fish between one and two years old bear 2 offspring each, and fish between two and three years old bear 1 offspring each. So we have that: $x_1(t+1) = 2x_2(t) + x_3(t)$.
- The number of fish between one and two years old at year $t+1$ is equal to 60% of the young fish at year t, so $x_2(t+1) = 0.6x_1(t)$.
- Similarly, the number of old fish at year $t+1$ is equal to 70% of the fish between one and two years old at year t. So $x_3(t+1) = 0.7x_2(t)$.

As a result, the population dynamics model is

$$\underbrace{\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \end{bmatrix}}_{x(t+1)} = \underbrace{\begin{bmatrix} 0 & 2 & 1 \\ 0.6 & 0 & 0 \\ 0 & 0.7 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}}_{x(t)}$$

5. *Another formula for the matrix exponential.* You might remember that for any complex number $a \in \mathbf{C}$, $e^a = \lim_{k\to\infty}(1 + a/k)^k$. You will establish the matrix analog: for any $A \in \mathbf{R}^{n\times n}$,

$$e^A = \lim_{k\to\infty}(I + A/k)^k.$$

To simplify things, you can assume A is diagonalizable. *Hint:* diagonalize. *Solution:*
Assuming $A \in \mathbf{R}^{k\times k}$ is diagonalizable, there exists an invertible matrix $T \in \mathbf{R}^{n\times n}$ such that $A = T\mathbf{diag}(\lambda_1, \ldots, \lambda_n)T^{-1}$ where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A. Therefore

$$\begin{aligned} (I + A/k)^k &= (TT^{-1} + T\mathbf{diag}(\lambda_1/k, \ldots, \lambda_n/k)T^{-1})^k \\ &= \left(T(I + \mathbf{diag}(\lambda_1/k, \ldots, \lambda_n/k))T^{-1}\right)^k \\ &= T(I + \mathbf{diag}(\lambda_1/k, \ldots, \lambda_n/k))^k T^{-1}. \end{aligned}$$

But $(I + \mathbf{diag}(\lambda_1/k, \ldots, \lambda_n/k))$ is diagonal and therefore its kth power is simply a diagonal matrix with diagonal entries equal to the kth power of the diagonal entries of $(I + \mathbf{diag}(\lambda_1/k, \ldots, \lambda_n/k))$. Thus

$$(I + A/k)^k = T\mathbf{diag}((1 + \lambda_1/k)^k, \ldots, (1 + \lambda_n/k)^k)T^{-1}$$

and taking the limit as $k \to \infty$ gives

$$\begin{aligned} \lim_{k\to\infty}(I + A/k)^k &= \lim_{k\to\infty} T\mathbf{diag}((1 + \lambda_1/k)^k, \ldots, (1 + \lambda_n/k)^k)T^{-1} \\ &= T\mathbf{diag}(\lim_{k\to\infty}(1 + \lambda_1/k)^k, \ldots, \lim_{k\to\infty}(1 + \lambda_n/k)^k)T^{-1} \\ &= T\mathbf{diag}(e^{\lambda_1}, \ldots, e^{\lambda_n})T^{-1} \\ &= e^A, \end{aligned}$$

and we are done.