

EE263 homework 6

1. *Optimal choice of initial temperature profile.* We consider a thermal system described by an n -element finite-element model. The elements are arranged in a line, with the temperature of element i at time t denoted $T_i(t)$. Temperature is measured in degrees Celsius above ambient; negative $T_i(t)$ corresponds to a temperature below ambient. The dynamics of the system are described by

$$c_1 \dot{T}_1 = -a_1 T_1 - b_1 (T_1 - T_2),$$

$$c_i \dot{T}_i = -a_i T_i - b_i (T_i - T_{i+1}) - b_{i-1} (T_i - T_{i-1}), \quad i = 2, \dots, n-1,$$

and

$$c_n \dot{T}_n = -a_n T_n - b_{n-1} (T_n - T_{n-1}).$$

where $c \in \mathbf{R}^n$, $a \in \mathbf{R}^n$, and $b \in \mathbf{R}^{n-1}$ are given and are all positive.

We can interpret this model as follows. The parameter c_i is the heat capacity of element i , so $c_i \dot{T}_i$ is the net heat flow into element i . The parameter a_i gives the thermal conductance between element i and the environment, so $a_i T_i$ is the heat flow from element i to the environment (*i.e.*, the direct heat loss from element i .) The parameter b_i gives the thermal conductance between element i and element $i+1$, so $b_i (T_i - T_{i+1})$ is the heat flow from element i to element $i+1$. Finally, $b_{i-1} (T_i - T_{i-1})$ is the heat flow from element i to element $i-1$.

The goal of this problem is to choose the initial temperature profile, $T(0) \in \mathbf{R}^n$, so that $T(t^{\text{des}}) \approx T^{\text{des}}$. Here, $t^{\text{des}} \in \mathbf{R}$ is a specific time when we want the temperature profile to closely match $T^{\text{des}} \in \mathbf{R}^n$. We also wish to satisfy a constraint that $\|T(0)\|$ should not be too large.

To formalize these requirements, we use the objective $(1/\sqrt{n})\|T(t^{\text{des}}) - T^{\text{des}}\|$ and the constraint $(1/\sqrt{n})\|T(0)\| \leq T^{\text{max}}$. The first expression is the RMS temperature deviation, at $t = t^{\text{des}}$, from the desired value, and the second is the RMS temperature deviation from ambient at $t = 0$. T^{max} is the (given) maximum initial RMS temperature value.

- (a) Explain how to find $T(0)$ that minimizes the objective while satisfying the constraint.
 - (b) Solve the problem instance with the values of n , c , a , b , t^{des} , T^{des} and T^{max} defined in the file `temp_prof_data.m`.
Plot, on one graph, your $T(0)$, $T(t^{\text{des}})$ and T^{des} . Give the RMS temperature error $(1/\sqrt{n})\|T(t^{\text{des}}) - T^{\text{des}}\|$, and the RMS value of initial temperature $(1/\sqrt{n})\|T(0)\|$.
2. *Positive quadrant invariance.* We consider a system $\dot{x} = Ax$ with $x(t) \in \mathbf{R}^2$ (although the results of this problem can be generalized to systems of higher dimension). We say the system is *positive quadrant invariant* (PQI) if whenever $x_1(T) \geq 0$ and $x_2(T) \geq 0$, we have $x_1(t) \geq 0$ and $x_2(t) \geq 0$ for all $t \geq T$. In other words, if the state starts inside (or enters) the positive (*i.e.*, first) quadrant, then the state remains indefinitely in the positive quadrant.
- (a) Find the precise conditions on A under which the system $\dot{x} = Ax$ is PQI. Try to express the conditions in the simplest form.
 - (b) *True or False:* if $\dot{x} = Ax$ is PQI, then the eigenvalues of A are real.
3. *Properties of the matrix exponential.*
- (a) Show that $e^{A+B} = e^A e^B$ if A and B commute, *i.e.*, $AB = BA$.
 - (b) Carefully show that $\frac{d}{dt} e^{At} = A e^{At} = e^{At} A$.

4. *A simple population model.* We consider a certain population of fish (say) each (yearly) season. $x(t) \in \mathbf{R}^3$ will describe the population of fish at year $t \in \mathbf{Z}$, as follows:

- $x_1(t)$ denotes the number of fish less than one year old
- $x_2(t)$ denotes the number of fish between one and two years old
- $x_3(t)$ denotes the number of fish between two and three years

(We will ignore the fact that these numbers are integers.) The population evolves from year t to year $t + 1$ as follows.

- The number of fish less than one year old in the next year ($t + 1$) is equal to the total number of offspring born during the current year. Fish that are less than one year old in the current year (t) bear no offspring. Fish that are between one and two years old in the current year (t) bear an average of 2 offspring each. Fish that are between two and three years old in the current year (t) bear an average of 1 offspring each.
- 40% of the fish less than one year old in the current year (t) die; the remaining 60% live on to be between one and two years old in the next year ($t + 1$).
- 30% of the one-to-two year old fish in the current year die, and 70% live on to be two-to-three year old fish in the next year.
- All of the two-to-three year old fish in the current year die.

Express the population dynamics as a discrete autonomous linear system with state $x(t)$, *i.e.*, in the form $x(t + 1) = Ax(t)$. **Remark:** this example is silly, but more sophisticated population dynamics models are very useful and widely used.

5. *Another formula for the matrix exponential.* You might remember that for any complex number $a \in \mathbf{C}$, $e^a = \lim_{k \rightarrow \infty} (1 + a/k)^k$. You will establish the matrix analog: for any $A \in \mathbf{R}^{n \times n}$,

$$e^A = \lim_{k \rightarrow \infty} (I + A/k)^k.$$

To simplify things, you can assume A is diagonalizable. *Hint:* diagonalize.