EE263 homework 4

1. *Orthogonal matrices.*

   (a) Show that if U and V are orthogonal, then so is UV.

   (b) Show that if U is orthogonal, then so is U^{-1}.

   (c) Suppose that $U \in \mathbf{R}^{2\times 2}$ is orthogonal. Show that U is either a rotation or a reflection. Make clear how you decide whether a given orthogonal U is a rotation or reflection.

Solution.

(a) To prove that UV is orthogonal we have to show that $(UV)^T(UV) = I$ given $U^TU = I$ and $V^TV = I$. We have

$$\begin{aligned}(UV)^T(UV) &= V^TU^TUV \\ &= V^TV \qquad \text{(since } U^TU = I) \\ &= I \qquad \text{(since } V^TV = I)\end{aligned}$$

and we are done.

(b) Since U is square and orthogonal we have $U^{-1} = U^T$ and therefore by taking inverses of both sides $U = (U^T)^{-1}$ or equivalently $U = (U^{-1})^T$ (the inverse and transpose operations commute.) But $U^TU = I$ and by substitution $U^{-1}(U^{-1})^T = I$. Since U^{-1} is square this also implies that $(U^{-1})^TU^{-1} = I$ so U^{-1} is orthogonal.

(c) Suppose that $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{R}^{2\times 2}$ is orthogonal. This is true if and only if

- columns of U are of unit length, *i.e.*, $a^2 + c^2 = 1$ and $b^2 + d^2 = 1$,
- columns of U are orthogonal, *i.e.*, $ab + cd = 0$.

Since $a^2 + c^2 = 1$ we can take a and c as the cosine and sine of an angle α respectively, *i.e.*, $a = \cos\alpha$ and $c = \sin\alpha$. For a similar reason, we can take $b = \sin\beta$ and $d = \cos\beta$. Now $ab + cd = 0$ becomes

$$\cos\alpha\sin\beta + \sin\alpha\cos\beta = 0$$

or

$$\sin(\alpha + \beta) = 0.$$

The sine of an angle is zero if and only if the angle is an integer multiple of π. So $\alpha + \beta = k\pi$ or $\beta = k\pi - \alpha$ with $k \in \mathbf{Z}$. Therefore

$$U = \begin{bmatrix} \cos\alpha & \sin(k\pi - \alpha) \\ \sin\alpha & \cos(k\pi - \alpha) \end{bmatrix}.$$

Now two things can happen:

- k is even so $\sin(k\pi - \alpha) = -\sin\alpha$ and $\cos(k\pi - \alpha) = \cos\alpha$, and therefore

$$U = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

  Clearly, from the lecture notes, this represents a rotation. Note that in this case $\det U = \cos^2\alpha + \sin^2\alpha = 1$.

- k is odd so $\sin(k\pi - \alpha) = \sin\alpha$ and $\cos(k\pi - \alpha) = -\cos\alpha$, and therefore

$$U = \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix}.$$

  From the lecture notes, this represents a reflection. The determinant in this case is $\det U = -\cos^2\alpha - \sin^2\alpha = -1$.

Therefore we have shown that any orthogonal matrix in $\mathbf{R}^{2\times 2}$ is either a rotation or reflection whether its determinant is $+1$ or -1 respectively.

2. *Projection matrices.* A matrix $P \in \mathbf{R}^{n\times n}$ is called a *projection matrix* if $P = P^T$ and $P^2 = P$.

   (a) Show that if P is a projection matrix then so is $I - P$.

   (b) Suppose that the columns of $U \in \mathbf{R}^{n\times k}$ are orthonormal. Show that UU^T is a projection matrix. (Later we will show that the converse is true: every projection matrix can be expressed as UU^T for some U with orthonormal columns.)

   (c) Suppose $A \in \mathbf{R}^{n\times k}$ is full rank, with $k \le n$. Show that $A(A^TA)^{-1}A^T$ is a projection matrix.

   (d) If $S \subseteq \mathbf{R}^n$ and $x \in \mathbf{R}^n$, the point y in S closest to x is called the *projection of x on S*. Show that if P is a projection matrix, then $y = Px$ is the projection of x on $\mathcal{R}(P)$. (Which is why such matrices are called projection matrices ...)

Solution.

(a) To show that $I - P$ is a projection matrix we need to check two properties:

   i. $I - P = (I - P)^T$

   ii. $(I - P)^2 = I - P$.

   The first one is easy: $(I - P)^T = I - P^T = I - P$ because $P = P^T$ (P is a projection matrix.) The show the second property we have

$$\begin{aligned} (I - P)^2 &= I - 2P + P^2 \\ &= I - 2P + P \qquad \text{(since } P = P^2\text{)} \\ &= I - P \end{aligned}$$

   and we are done.

(b) Since the columns of U are orthonormal we have $U^TU = I$. Using this fact it is easy to prove that UU^T is a projection matrix, *i.e.*, $(UU^T)^T = UU^T$ and $(UU^T)^2 = UU^T$. Clearly, $(UU^T)^T = (U^T)^TU^T = UU^T$ and

$$\begin{aligned} (UU^T)^2 &= (UU^T)(UU^T) \\ &= U(U^TU)U^T \\ &= UU^T \qquad \text{(since } U^TU = I\text{).} \end{aligned}$$

(c) First note that $\left(A(A^TA)^{-1}A^T\right)^T = A(A^TA)^{-1}A^T$ because

$$\begin{aligned} \left(A(A^TA)^{-1}A^T\right)^T &= (A^T)^T\left((A^TA)^{-1}\right)^T A^T \\ &= A\left((A^TA)^T\right)^{-1} A^T \\ &= A(A^TA)^{-1}A^T. \end{aligned}$$

Also $\left(A(A^TA)^{-1}A^T\right)^2 = A(A^TA)^{-1}A^T$ because

$$\begin{aligned}
\left(A(A^TA)^{-1}A^T\right)^2 &= \left(A(A^TA)^{-1}A^T\right)\left(A(A^TA)^{-1}A^T\right) \\
&= A\left((A^TA)^{-1}A^TA\right)(A^TA)^{-1}A^T \\
&= A(A^TA)^{-1}A^T \qquad \text{(since } (A^TA)^{-1}A^TA = I\text{)}.
\end{aligned}$$

(d) To show that Px is the projection of x on $\mathcal{R}(P)$ we verify that the "error" $x - Px$ is orthogonal to *any* vector in $\mathcal{R}(P)$. Since $\mathcal{R}(P)$ is nothing but the span of the columns of P we only need to show that $x - Px$ is orthogonal to the columns of P, or in other words, $P^T(x - Px) = 0$. But

$$\begin{aligned}
P^T(x - Px) &= P(x - Px) \qquad \text{(since } P = P^T\text{)} \\
&= Px - P^2x \\
&= 0 \qquad \text{(since } P^2 = P\text{)}
\end{aligned}$$

and we are done.

3. *Householder reflections.* A *Householder matrix* is defined as

$$Q = I - 2uu^T,$$

where $u \in \mathbf{R}^n$ is normalized, that is, $u^Tu = 1$.

(a) Show that Q is orthogonal.

(b) Show that $Qu = -u$. Show that $Qv = v$, for any v such that $u^Tv = 0$. Thus, multiplication by Q gives reflection through the plane with normal vector u.

(c) Show that $\det Q = -1$.

(d) Given a vector $x \in \mathbf{R}^n$, find a unit-length vector u for which Qx lies on the line through e_1. *Hint:* Try a u of the form $u = v/\|v\|$, with $v = x + \alpha e_1$ (find the appropriate α and show that such a u works ...) Compute such a u for $x = (3, 2, 4, 1, 5)$. Apply the corresponding Householder reflection to x to find Qx.

Note: Multiplication by an orthogonal matrix has very good numerical properties, in the sense that it does not accumulate much roundoff error. For this reason, Householder reflections are used as building blocks for fast, numerically sound algorithms.

Solution.

(a)

$$\begin{aligned}
Q^TQ &= (I - 2uu^T)^T(I - 2uu^T) \\
&= (I - 2uu^T)(I - 2uu^T) \\
&= I - 2uu^T - 2uu^T + 4uu^Tuu^T \\
&= I - 2uu^T - 2uu^T + 4uu^T \qquad \text{using } u^Tu = 1 \\
&= I \qquad \text{so Q is orthogonal}
\end{aligned}$$

(b)

$$\begin{aligned}
Qu &= u - 2uu^Tu = u - 2u = -u \qquad \text{using } u^Tu = 1 \\
Qv &= v - 2uu^tv = v \qquad \text{using } u^Tv = 0
\end{aligned}$$

(c) We know $\det(Q) = \prod_{i=1}^{n} \lambda_i$. Since Q is symmetric, all eigenvalues are real and we can construct an orthonormal eigenvector basis. From parts (a) and (b), u is an eigenvector with associated eigenvalue -1, and any vector v orthogonal to u is an eigenvector with associated eigenvalue 1. The nullspace of u^T has dimension $n-1$, so we can construct an orthogonal eigenbasis with all eigenvalues 1 except for the -1 eigenvalue with eigenvector u. Thus the product of the eigenvalues is $-1 = \det(Q)$.

(d) We follow the hint and choose $u = (x + \alpha e_1)/\|x + \alpha e_1\|$. Then

$$
\begin{aligned}
Q &= I - 2\frac{(x+\alpha e_1)(x+\alpha e_1)^T}{(x+\alpha e_1)^T(x+\alpha e_1)} \\
&= I - 2\frac{x(x^T + \alpha e_1^T) + \alpha e_1(x^T + \alpha e_1^T)}{x^Tx + \alpha e_1^T x + \alpha x^T e_1 + \alpha^2 e_1^T e_1} \\
Qx &= x - 2\frac{x(\|x\|^2 + \alpha e_1^T x) + e_1(\alpha\|x\|^2 + \alpha^2 x_1}{\|x\|^2 + 2\alpha x_1 + \alpha^2} \\
&= x - \frac{2\|x\|^2 + 2\alpha x_1}{\|x\|^2 + 2\alpha x_1 + \alpha^2} x - 2\alpha \frac{\|x\|^2 + \alpha x_1}{\|x\|^2 + 2\alpha x_1 + \alpha^2} e_1 \\
&= \underbrace{\left(1 - \frac{2\|x\|^2 + 2\alpha x_1}{\|x\|^2 + 2\alpha x_1 + \alpha^2}\right)}_{\text{Need this zero}} x - 2\alpha \frac{\|x\|^2 + \alpha x_1}{\|x\|^2 + 2\alpha x_1 + \alpha^2} e_1
\end{aligned}
$$

We can achieve this by choosing $\alpha = \pm\|x\|$. This leads to $Qx = \mp\|x\|e_1$ (which makes sense ... Q should always preserve norm). Some people used a geometric argument here as well, and this can make the solution a lot neater if it's well presented. The idea is to find a reflection plane that reflects the given vector onto the e_1 axis (there are two possibilities, for negative and positive parts of the e_1 axis), and u is then a unit vector orthogonal to this plane.

4. *Interpolation with rational functions.* In this problem we consider a function $f : \mathbf{R} \to \mathbf{R}$ of the form

$$f(x) = \frac{a_0 + a_1 x + \cdots + a_m x^m}{1 + b_1 x + \cdots + b_m x^m},$$

where $a_0, \ldots, a_m$, and $b_1, \ldots, b_m$ are parameters, with either $a_m \neq 0$ or $b_m \neq 0$. Such a function is called a *rational function of degree m*. We are given data points $x_1, \ldots, x_N \in \mathbf{R}$ and $y_1, \ldots, y_N \in \mathbf{R}$, where $y_i = f(x_i)$. The problem is to find a rational function of smallest degree that is consistent with this data. In other words, you are to find m, which should be as small as possible, and $a_0, \ldots, a_m$, $b_1, \ldots, b_m$, which satisfy $f(x_i) = y_i$. Explain how you will solve this problem, and then carry out your method on the problem data given in `ri_data.m`. (This contains two vectors, `x` and `y`, that give the values $x_1, \ldots, x_N$, and $y_1, \ldots, y_N$, respectively.) Give the value of m you find, and the coefficients $a_0, \ldots, a_m$, $b_1, \ldots, b_m$. Please show us your verification that $y_i = f(x_i)$ holds (possibly with some small numerical errors).

Solution. The interpolation condition $f(x_i) = y_i$ is

$$f(x_i) = \frac{a_0 + a_1 x_i + \cdots + a_m x_i^m}{1 + b_1 x_i + \cdots + b_m x_i^m} = y_i, \quad i = 1, \ldots, N.$$

This is a set of complicated nonlinear functions of the coefficient vectors a and b. If we multiply out by the denominator, we get

$$y_i(1 + b_1 x_i + \cdots + b_m x_i^m) - (a_0 + a_1 x_i + \cdots + a_m x_i^m) = 0, \quad i = 1, \ldots, N.$$

These equations are *linear* in a and b. We can write these equations in matrix form as

$$G \begin{bmatrix} a \\ b \end{bmatrix} = y, \tag{1}$$

where

$$a = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_m \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},$$

and

$$G = \begin{bmatrix} 1 & x_1 & \cdots & x_1^m & -y_1 x_1 & -y_1 x_1^2 & \cdots & -y_1 x_1^m \\ 1 & x_2 & \cdots & x_2^m & -y_2 x_2 & -y_2 x_2^2 & \cdots & -y_2 x_2^m \\ 1 & x_3 & \cdots & x_3^m & -y_3 x_3 & -y_3 x_3^2 & \cdots & -y_3 x_3^m \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & x_N & \cdots & x_N^m & -y_N x_N & -y_N x_N^2 & \cdots & -y_N x_N^m \end{bmatrix}.$$

Thus, we can interpolate the data if and only if the equation (1) has a solution. Our problem is to find the smallest m for which these linear equations can be solved, or, equivalently, $y \in \mathcal{R}(G)$. We can do this by finding the smallest value of m for which

$$\mathbf{Rank}(G) = \mathbf{Rank}([G\ y]).$$

Then we can find a set of coefficients by solving the equation (1) for a and b. The following Matlab code carries out this method.

```
clear all
close all
rat_int_data
for m=1:20 %we sweep over different values of m
G=ones(N,1);
for i=1:m;
G=[G x.^i];
end
for i=1:m
G=[G -x.^i.*y];
end
if rank(G)== rank([G y])
break;
end
end
ab=G\y;
a=ab(1:m+1);
b=ab(m+2:2*m+1);
m
a
b
```

Matlab produces the following output:

```
m =
5
```

```
a =
0.2742
1.0291
1.2906
-5.8763
-2.6738
6.6845
b =
-1.2513
-6.5107
3.2754
17.3797
6.6845
```

Thus, we find that $m = 5$ is the lowest order rational function that interpolates the data, and a rational function that interpolates the data is given by

$$f(x) = \frac{0.2742 + 1.0291x + 1.2906x^2 - 5.8763x^3 - 2.6738x^4 + 6.6845x^5}{1 - 1.2513x - 6.5107x^2 + 3.2754x^3 + 17.3797x^4 + 6.6845x^5}$$

(we have truncated the coefficients to shorten the formula). We now verify that this expression interpolates the given points.

```
num=zeros(N,1);
for i=1:m+1
num=a(i)*x.^(i-1)+num;
end
den=ones(N,1);
for i=1:m
den=b(i)*x.^i+den;
end
f=num./den;
err=norm(f-y)
```

Matlab produces the following output

```
err =
7.7649e-14.
```

This shows that the output is interpolated up to numerical precision.

5. *Finding a basis for the intersection of ranges.*

   (a) Suppose you are given two matrices, $A \in \mathbf{R}^{n \times p}$ and $B \in \mathbf{R}^{n \times q}$. Explain how you can find a matrix $C \in \mathbf{R}^{n \times r}$, with independent columns, for which

   $$\mathcal{R}(C) = \mathcal{R}(A) \cap \mathcal{R}(B).$$

   This means that the columns of C are a basis for $\mathcal{R}(A) \cap \mathcal{R}(B)$.

   Hint: begin by showing that if S_1 and S_2 are subspaces of $\mathbf{R}^n$, then $(S_1 \cap S_2)^\perp = S_1^\perp + S_2^\perp$, where the notation "+" is overloaded for subspaces to mean: $S_1 + S_2 = \{x_1 + x_2 \mid x_1 \in S_1,\, x_2 \in S_2\}$. Note that $S_1 + S_2$ is again a subspace.

(b) Carry out the method described in part (a) for the particular matrices A and B defined in `intersect_range_data.m`. Be sure to give us your matrix C, as well as the Matlab (or other) code that generated it. Verify that $\mathcal{R}(C) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(C) \subseteq \mathcal{R}(B)$, by showing that each column of C is in the range of A, and also in the range of B.

Please carefully separate your answers to part (a) (the general case) and part (b) (the specific case).

Solution.

(a) We know that

$$\mathcal{R}(A) = \mathcal{N}(A^T)^\perp.$$

This means that any y in $\mathcal{R}(A)$ is perpendicular to all vectors in the $\mathcal{N}(A^T)$; and any vector which is perpendicular to all vectors in $\mathcal{N}(A^T)$, must be in $\mathcal{R}(A)$. We will show that

$$\mathcal{R}(A) \cap \mathcal{R}(B) = \left(\mathcal{N}(A^T) + \mathcal{N}(B^T)\right)^\perp.$$

Let y be a vector in $\mathcal{R}(A) \cap \mathcal{R}(B)$. Then $y = Ax_a$, for some x_a and $y = Bx_b$, for some x_b. Let v be any vector in the $\mathcal{N}(A^T) + \mathcal{N}(B^T)$. Then $v = v_a + v_b$ for some $v_a \in \mathcal{N}(B^T), v_b \in \mathcal{N}(B^T)$. Then we have,

$$y^T v = y^T v_a + y^T v_b = x_a^T A^T v_a + x_b^T B^T v_b = x_a^T (A^T v_a) + x_b^T (B^T v_b) = 0.$$

Thus $y \perp \left(\mathcal{N}(A^T) + \mathcal{N}(B^T)\right)$. Since any vector in $(\mathcal{R}(A) \cap \mathcal{R}(B))$ is perpendicular to every vector in $\left(\mathcal{N}(A^T) + \mathcal{N}(B^T)\right)$,

$$\mathcal{R}(A) \cap \mathcal{R}(B) \subseteq \left(\mathcal{N}(A^T) + \mathcal{N}(B^T)\right)^\perp.$$

Let y be a vector in $\left(\mathcal{N}(A^T) + \mathcal{N}(B^T)\right)^\perp$. Then y is perpendicular to all vectors in $\mathcal{N}(A^T)$ which means $y \in \mathcal{R}(A)$. Similarly y is perpendicular to all vectors in $\mathcal{N}(B^T)$ which means $y \in \mathcal{R}(B)$. Thus $y \in (\mathcal{R}(A) \cap \mathcal{R}(B))$ and we have,

$$\mathcal{R}(A) \cap \mathcal{R}(B) = \left(\mathcal{N}(A^T) + \mathcal{N}(B^T)\right)^\perp.$$

The full QR factorization of a matrix A is,

$$A = [Q_{1A}\ Q_{2A}] \begin{bmatrix} R_{1A} \\ 0 \end{bmatrix},$$

and $\mathcal{N}(A^T) = \mathcal{R}(Q_{2A})$. Similarly, let full QR factorization if a matrix B be

$$B = [Q_{1B}\ Q_{2B}] \begin{bmatrix} R_{1B} \\ 0 \end{bmatrix},$$

and hence $\mathcal{N}(B^T) = \mathcal{R}(Q_{2B})$. Then,

$$\mathcal{N}(A^T) + \mathcal{N}(B^T) = \mathcal{R}(Q_{2A}) + \mathcal{R}(Q_{2B}) = \mathcal{R}(D),$$

where $D = [Q_{2A}\ Q_{2B}]$. Now,

$$\mathcal{R}(A) \cap \mathcal{R}(B) = \left(\mathcal{N}(A^T) + \mathcal{N}(B^T)\right)^\perp = \mathcal{R}(D)^\perp = \mathcal{N}(D^T).$$

So we find the QR factorization of D. Let the QR factorization be

$$D = [Q_{1D}\ Q_{2D}] \begin{bmatrix} R_{1D} \\ 0 \end{bmatrix},$$

and then $C = Q_{2D}$ as $\mathcal{N}(D^T) = \mathcal{R}(Q_{2D}) = \mathcal{R}(C)$. Thus we have the matrix C such that $\mathcal{R}(C) = \mathcal{R}(A) \cap \mathcal{R}(B)$.

(b) The following Matlab code gives the required matrix C and the dimension of $\mathcal{R}(C)$.

```
clear ;
intersect_range_data;
Q_2A = null(A');
Q_2B = null(B');
D = [Q_2A Q_2B];
C = null(D')
rC = rank(C)
>>
C =
-0.3365   -0.2349    0.3581
0.2927   -0.4471   -0.0277
-0.6691    0.0460    0.0131
0.1963    0.3655    0.2581
0.3599   -0.1406   -0.1416
-0.0929    0.1880   -0.5108
0.1967    0.4497    0.3712
0.2019   -0.5007    0.0800
0.2901    0.2292    0.2283
-0.1140   -0.2208    0.5718
rC = 3
```

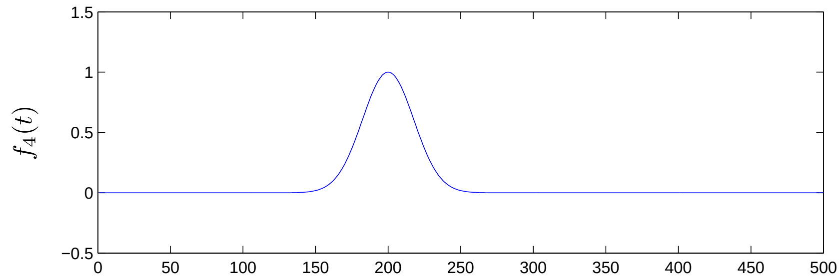
Show $\mathcal{R}(C) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(C) \subseteq \mathcal{R}(B)$.

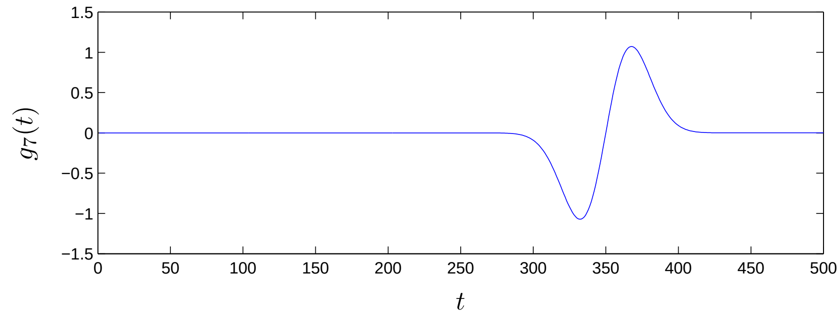
```
rA = rank(A)
rAC = rank([A C])
rB = rank(B)
rBC = rank([B C])
>>
rA = 6
rAC = 6
rB = 5
rBC = 5
```

6. *Signal estimation using least-squares.* This problem concerns discrete-time signals defined for $t = 1, \ldots, 500$. We'll represent these signals by vectors in $\mathbf{R}^{500}$, with the index corresponding to the time. We are given a noisy measurement $y_{\text{meas}}(1), \ldots, y_{\text{meas}}(500)$, of a signal $y(1), \ldots, y(500)$ that is thought to be, at least approximately, a linear combination of the 22 signals

$$f_k(t) = e^{-(t-50k)^2/25^2}, \qquad g_k(t) = \left(\frac{t-50k}{10}\right) e^{-(t-50k)^2/25^2},$$

where $t = 1, \ldots, 500$ and $k = 0, \ldots, 10$. Plots of f_4 and g_7 (as examples) are shown below.





As our estimate of the original signal, we will use the signal $\hat{y} = (\hat{y}(1), \ldots, \hat{y}(500))$ in the span of $f_0, \ldots, f_{10}, g_0, \ldots, g_{10}$, that is closest to $y_{\rm meas} = (y_{\rm meas}(1), \ldots, y_{\rm meas}(500))$ in the RMS (root-mean-square) sense. Explain how to find $\hat{y}$, and carry out your method on the signal $y_{\rm meas}$ given in `sig_est_data.m` on the course web site. Plot $y_{\rm meas}$ and $\hat{y}$ on the same graph. Plot the residual (the difference between these two signals) on a different graph, and give its RMS value.

Solution. We'll form the estimated signal as a linear combination of $f_0, \ldots, f_{10}, g_0, \ldots, g_{10}$,

$$\hat{y} = x_1 f_0 + x_2 f_1 + \cdots + x_{11} f_{10} + x_{12} g_0 + \cdots + x_{22} g_{10}.$$

(Here we are representing the signals as vectors in $\mathbf{R}^{500}$.) We can write this in matrix form as $\hat{y} = Ax$, where

$$A = [f_0 \ f_1 \ \cdots \ f_{10} \ g_0 \ \cdots \ g_{10}] \in \mathbf{R}^{500\times 22}.$$

The coefficients x are chosen to minimize the RMS deviation between $\hat{y}$ and $y_{\rm meas}$, which is the same as minimizing the norm of the difference. The matrix A is full rank (*i.e.*, 22), so the best coefficients are given by

$$x_{\rm ls} = (A^T A)^{-1} A^T y_{\rm meas}.$$

Our estimate of the original signal is

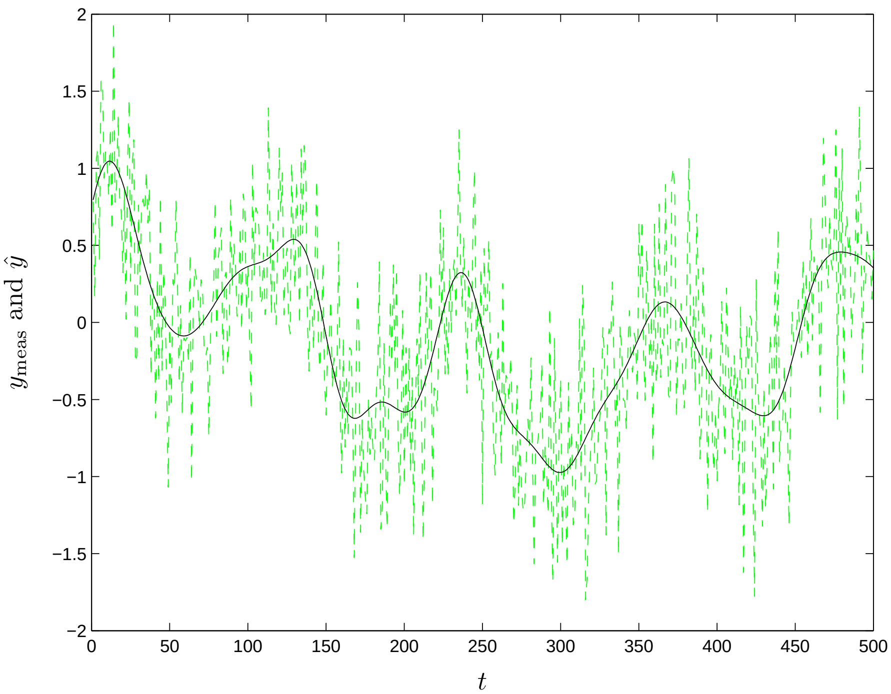
$$\hat{y} = A x_{\rm ls} = A (A^T A)^{-1} A^T y_{\rm meas}.$$

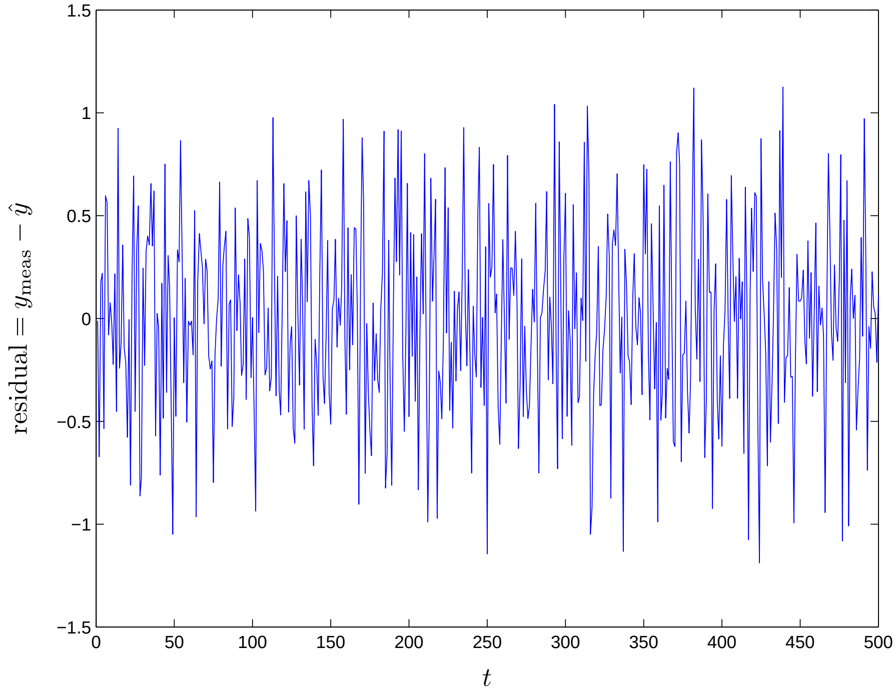
The following Matlab code implements this estimation method.

```
sig_est_data;
nfcts = 22; ydim = 500;
t = 1:ydim;
A = zeros(ydim,nfcts);
for k = 1:nfcts/2
fk = exp(-(t-50*(k-1)).^2/25^2);
gk = (t-50*(k-1))/10.*exp(-(t-50*(k-1)).^2/25^2);
```

```
A(:,k) = fk'; A(:,k+11) = gk';
end
yhat = A*(A\ymeas);
residual = ymeas-yhat;
RMS = 1/sqrt(500)*norm(residual);
figure(1); plot(t,ymeas,'g--',t,yhat,'k');
xlabel('time'); ylabel('fit');
figure(2); plot(t,ymeas-yhat);
xlabel('time'); ylabel('residual');
>> rank(A) = 22
>> RMS = 0.4596
```

The figure below shows the measured signal $y_{\rm meas}$ and the estimated signal $\hat{y}$. The RMS value of the residual is 0.4596. The next figure shows the residual.




1.5
1
0.5
0
−0.5
−1
−1.5
0
50
100
150
200
250
300
350
400
450
500
residual = $y_{\mathrm{meas}} - \hat{y}$
t