EE263 homework 4

1. *Orthogonal matrices.*
   - (a) Show that if U and V are orthogonal, then so is UV.
   - (b) Show that if U is orthogonal, then so is U^{-1}.
   - (c) Suppose that $U \in \mathbf{R}^{2\times 2}$ is orthogonal. Show that U is either a rotation or a reflection. Make clear how you decide whether a given orthogonal U is a rotation or reflection.

2. *Projection matrices.* A matrix $P \in \mathbf{R}^{n\times n}$ is called a *projection matrix* if $P = P^T$ and $P^2 = P$.
   - (a) Show that if P is a projection matrix then so is $I - P$.
   - (b) Suppose that the columns of $U \in \mathbf{R}^{n\times k}$ are orthonormal. Show that UU^T is a projection matrix. (Later we will show that the converse is true: every projection matrix can be expressed as UU^T for some U with orthonormal columns.)
   - (c) Suppose $A \in \mathbf{R}^{n\times k}$ is full rank, with $k \le n$. Show that $A(A^TA)^{-1}A^T$ is a projection matrix.
   - (d) If $S \subseteq \mathbf{R}^n$ and $x \in \mathbf{R}^n$, the point y in S closest to x is called the *projection of x on S*. Show that if P is a projection matrix, then $y = Px$ is the projection of x on $\mathcal{R}(P)$. (Which is why such matrices are called projection matrices . . .)

3. *Householder reflections.* A *Householder matrix* is defined as

   $$Q = I - 2uu^T,$$

   where $u \in \mathbf{R}^n$ is normalized, that is, $u^Tu = 1$.
   - (a) Show that Q is orthogonal.
   - (b) Show that $Qu = -u$. Show that $Qv = v$, for any v such that $u^Tv = 0$. Thus, multiplication by Q gives reflection through the plane with normal vector u.
   - (c) Show that $\det Q = -1$.
   - (d) Given a vector $x \in \mathbf{R}^n$, find a unit-length vector u for which Qx lies on the line through e_1. *Hint:* Try a u of the form $u = v/\|v\|$, with $v = x + \alpha e_1$ (find the appropriate α and show that such a u works . . .) Compute such a u for $x = (3, 2, 4, 1, 5)$. Apply the corresponding Householder reflection to x to find Qx.

   Note: Multiplication by an orthogonal matrix has very good numerical properties, in the sense that it does not accumulate much roundoff error. For this reason, Householder reflections are used as building blocks for fast, numerically sound algorithms.

4. *Interpolation with rational functions.* In this problem we consider a function $f : \mathbf{R} \to \mathbf{R}$ of the form

   $$f(x) = \frac{a_0 + a_1x + \cdots + a_mx^m}{1 + b_1x + \cdots + b_mx^m},$$

   where $a_0, \ldots, a_m$, and $b_1, \ldots, b_m$ are parameters, with either $a_m \ne 0$ or $b_m \ne 0$. Such a function is called a *rational function of degree m*. We are given data points $x_1, \ldots, x_N \in \mathbf{R}$ and $y_1, \ldots, y_N \in \mathbf{R}$, where $y_i = f(x_i)$. The problem is to find a rational function of smallest degree that is consistent with this data. In other words, you are to find m, which should be as small as possible, and $a_0, \ldots, a_m$, $b_1, \ldots, b_m$, which satisfy $f(x_i) = y_i$. Explain how you will solve this problem, and then carry out your method on the problem data given in `ri_data.m`. (This contains two vectors, `x` and `y`, that give the values $x_1, \ldots, x_N$, and $y_1, \ldots, y_N$, respectively.) Give the value of m you find, and the coefficients $a_0, \ldots, a_m$, $b_1, \ldots, b_m$. Please show us your verification that $y_i = f(x_i)$ holds (possibly with some small numerical errors).

5. *Finding a basis for the intersection of ranges.*

   (a) Suppose you are given two matrices, $A \in \mathbf{R}^{n\times p}$ and $B \in \mathbf{R}^{n\times q}$. Explain how you can find a matrix $C \in \mathbf{R}^{n\times r}$, with independent columns, for which

   $$\mathcal{R}(C) = \mathcal{R}(A) \cap \mathcal{R}(B).$$

   This means that the columns of C are a basis for $\mathcal{R}(A) \cap \mathcal{R}(B)$.

   Hint: begin by showing that if S_1 and S_2 are subspaces of $\mathbf{R}^n$, then $(S_1 \cap S_2)^\perp = S_1^\perp + S_2^\perp$, where the notation "+" is overloaded for subspaces to mean: $S_1 + S_2 = \{x_1 + x_2 \mid x_1 \in S_1,\ x_2 \in S_2\}$. Note that $S_1 + S_2$ is again a subspace.

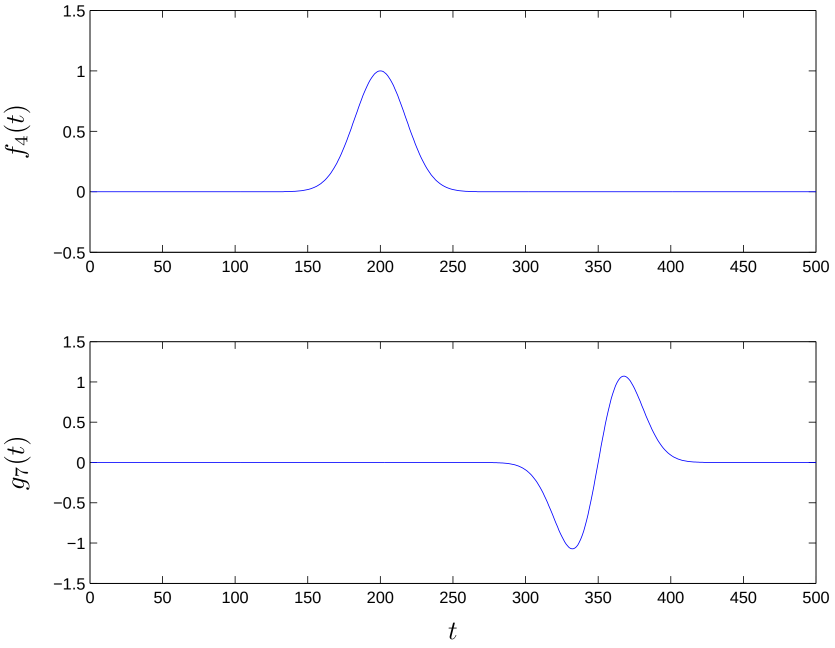
   (b) Carry out the method described in part (a) for the particular matrices A and B defined in `intersect_range_data.m`. Be sure to give us your matrix C, as well as the Matlab (or other) code that generated it. Verify that $\mathcal{R}(C) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(C) \subseteq \mathcal{R}(B)$, by showing that each column of C is in the range of A, and also in the range of B.

   Please carefully separate your answers to part (a) (the general case) and part (b) (the specific case).

6. *Signal estimation using least-squares.* This problem concerns discrete-time signals defined for $t = 1, \dots, 500$. We'll represent these signals by vectors in $\mathbf{R}^{500}$, with the index corresponding to the time. We are given a noisy measurement $y_{\rm meas}(1), \dots, y_{\rm meas}(500)$, of a signal $y(1), \dots, y(500)$ that is thought to be, at least approximately, a linear combination of the 22 signals

   $$f_k(t) = e^{-(t-50k)^2/25^2}, \qquad g_k(t) = \left(\frac{t-50k}{10}\right) e^{-(t-50k)^2/25^2},$$

   where $t = 1, \dots, 500$ and $k = 0, \dots, 10$. Plots of f_4 and g_7 (as examples) are shown below.



   As our estimate of the original signal, we will use the signal $\hat{y} = (\hat{y}(1), \dots, \hat{y}(500))$ in the span of $f_0, \dots, f_{10}, g_0, \dots, g_{10}$, that is closest to $y_{\rm meas} = (y_{\rm meas}(1), \dots, y_{\rm meas}(500))$ in the RMS (root-mean-square) sense. Explain how to find $\hat{y}$, and carry out your method on the signal $y_{\rm meas}$ given in `sig_est_data.m` on the course web site. Plot $y_{\rm meas}$ and $\hat{y}$ on the same graph. Plot the residual (the difference between these two signals) on a different graph, and give its RMS value.