EE263 homework 3

1. *Linearizing range measurements.* Consider a single (scalar) measurement y of the distance or range of $x \in \mathbf{R}^n$ to a fixed point or beacon at a, *i.e.*, $y = \|x - a\|$.

   (a) Show that the linearized model near x_0 can be expressed as $\delta y = k^T \delta x$, where k is the unit vector (*i.e.*, with length one) pointing from a to x_0. Derive this analytically, and also draw a picture (for $n = 2$) to demonstrate it.

   (b) Consider the error e of the linearized approximation, *i.e.*,

   $$e = \|x_0 + \delta x - a\| - \|x_0 - a\| - k^T \delta x.$$

   The relative error of the approximation is given by $\eta = e/\|x_0 - a\|$. We know, of course, that the absolute value of the relative error is very small provided δx is small. In many specific applications, it is possible and useful to make a stronger statement, for example, to derive a bound on how large the error can be. You will do that here. In fact you will prove that

   $$0 \le \eta \le \frac{\alpha^2}{2}$$

   where $\alpha = \|\delta x\|/\|x_0 - a\|$ is the relative size of δx. For example, for a relative displacement of $\alpha = 1\%$, we have $\eta \le 0.00005$, *i.e.*, the linearized model is accurate to about 0.005%. To prove this bound you can proceed as follows:

   - Show that $\eta = -1 + \sqrt{1 + \alpha^2 + 2\beta} - \beta$ where $\beta = k^T \delta x/\|x_0 - a\|$.
   - Verify that $|\beta| \le \alpha$.
   - Consider the function $g(\beta) = -1 + \sqrt{1 + \alpha^2 + 2\beta} - \beta$ with $|\beta| \le \alpha$. By maximizing and minimizing g over the interval $-\alpha \le \beta \le \alpha$ show that

   $$0 \le \eta \le \frac{\alpha^2}{2}.$$

2. *Halfspace.* Suppose $a, b \in \mathbf{R}^n$ are two given points. Show that the set of points in $\mathbf{R}^n$ that are closer to a than b is a halfspace, *i.e.*:

   $$\{x \mid \|x - a\| \le \|x - b\| \} = \{ x \mid c^T x \le d\}$$

   for appropriate $c \in \mathbf{R}^n$ and $d \in \mathbf{R}$. Give c and d explicitly, and draw a picture showing a, b, c, and the halfspace.

3. *Some properties of the product of two matrices.* For each of the following statements, either show that it is true, or give a (specific) counterexample.

   - If AB is full rank then A and B are full rank.
   - If A and B are full rank then AB is full rank.
   - If A and B have zero nullspace, then so does AB.
   - If A and B are onto, then so is AB.

   You can assume that $A \in \mathbf{R}^{m \times n}$ and $B \in \mathbf{R}^{n \times p}$. Some of the false statements above become true under certain assumptions on the dimensions of A and B. As a trivial example, all of the statements above are true when A and B are scalars, *i.e.*, $n = m = p = 1$. For each of the statements above, find conditions on n, m, and p that make them true. Try to find the most general conditions you can. You can give your conditions as inequalities involving n, m, and p, or you can use more informal language such as "A and B are both skinny."

4. *Some true/false questions.* Determine if the following statements are true or false. No justification or discussion is needed for your answers. What we mean by "true" is that the statement is true for all values of the matrices and vectors given. You can't assume anything about the dimensions of the matrices (unless it's explicitly stated), but you can assume that the dimensions are such that all expressions make sense. For example, the statement "$A+B = B+A$" is true, because no matter what the dimensions of A and B (which must, however, be the same), and no matter what values A and B have, the statement holds. As another example, the statement $A^2 = A$ is false, because there are (square) matrices for which this doesn't hold. (There are also matrices for which it does hold, *e.g.*, an identity matrix. But that doesn't make the statement true.)

   a. If all coefficients (*i.e.*, entries) of the matrix A are positive, then A is full rank.
   b. If A and B are onto, then $A + B$ must be onto.
   c. If A and B are onto, then so is the matrix $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$.
   d. If A and B are onto, then so is the matrix $\begin{bmatrix} A \\ B \end{bmatrix}$.
   e. If the matrix $\begin{bmatrix} A \\ B \end{bmatrix}$ is onto, then so are the matrices A and B.
   f. If A is full rank and skinny, then so is the matrix $\begin{bmatrix} A \\ B \end{bmatrix}$.

5. *Orthogonal complement of a subspace.* If $\mathcal{V}$ is a subspace of $\mathbf{R}^n$ we define $\mathcal{V}^\perp$ as the set of vectors orthogonal to every element in $\mathcal{V}$, *i.e.*,

$$\mathcal{V}^\perp = \{\, x \mid \langle x, y \rangle = 0, \ \forall y \in \mathcal{V} \,\}.$$

   (a) Verify that $\mathcal{V}^\perp$ is a subspace of $\mathbf{R}^n$.
   (b) Suppose $\mathcal{V}$ is described as the span of some vectors $v_1, v_2, \ldots, v_r$. Express $\mathcal{V}$ and $\mathcal{V}^\perp$ in terms of the matrix $V = \begin{bmatrix} v_1 & v_2 & \cdots & v_r \end{bmatrix} \in \mathbf{R}^{n \times r}$ using common terms (range, nullspace, transpose, etc.)
   (c) Show that every $x \in \mathbf{R}^n$ can be expressed uniquely as $x = v + v^\perp$ where $v \in \mathcal{V}$, $v^\perp \in \mathcal{V}^\perp$. *Hint:* let v be the projection of x on $\mathcal{V}$.
   (d) Show that $\dim \mathcal{V}^\perp + \dim \mathcal{V} = n$.
   (e) Show that $\mathcal{V} \subseteq \mathcal{U}$ implies $\mathcal{U}^\perp \subseteq \mathcal{V}^\perp$.

6. *Temperatures in a multi-core processor.* We are concerned with the temperature of a processor at two critical locations. These temperatures, denoted $T = (T_1, T_2)$ (in degrees C), are affine functions of the power dissipated by three processor cores, denoted $P = (P_1, P_2, P_3)$ (in W). We make 4 measurements. In the first, all cores are idling, and dissipate 10W. In the next three measurements, one of the processors is set to full power, 100W, and the other two are idling. In each experiment we measure and note the temperatures at the two critical locations.

P_1	P_2	P_3	T_1	T_2
10W	10W	10W	27°	29°
100W	10W	10W	45°	37°
10W	100W	10W	41°	49°
10W	10W	100W	35°	55°

Suppose we operate all cores at the same power, p. How large can we make p, without T_1 or T_2 exceeding 70°?

You must fully explain your reasoning and method, in addition to providing the numerical solution.