Reconstruction

We now discuss reconstruction, that is putting back together a two-dimensional distribution from a collection of images, each representing part of the desired signal. Specifically, we will now look at reconstruction of a two-D array from a series of 1-D integrated measurements -- projections of the desired distribution.

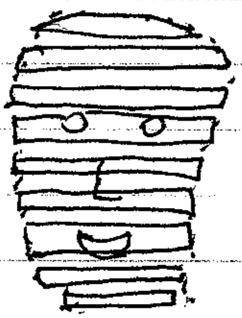
We have previously looked at one reconstruction problem, that of populating the u-v plane by diverse interferometer measurements. In that case each interferometer configuration yielded some autocorrelation islands, and the sum of several of these covered the plane more completely than a single antenna could.

Today's topic derives from applying similar thinking to to the problem of measuring projections rather than the full distribution. This was widely applied in medicine to x-ray tomography, but similar techniques have proven invaluable for many applications, such as seismic mapping of the inner Earth. Any time line integrals of a quantity are more readily available than the desired quantity itself tomographic reconstruction must be considered.

Example application - x-ray tomography

X-rays provide reasonably non-invasive imaging of the human body. For many applications, say for example bone fractures, simple projections of the density distribution suffice to help medical diagnoses. But for detailed measurements, such as mapping subtle variations in tissue density, resolving the full 3-D structure may be important.

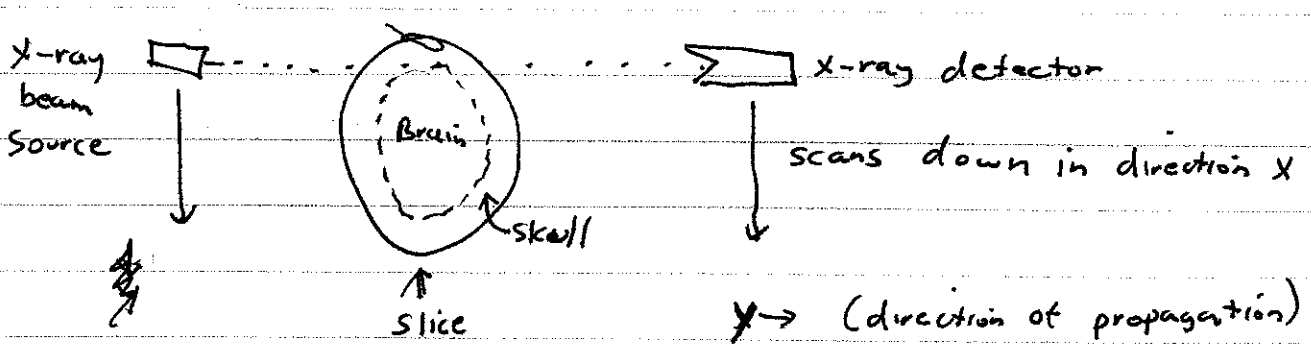
Consider an x-ray beam that can be moved in one dimension to measure "slices" of an object. If several slices at different locations are stacked, the 3-D distribution can be reconstructed.



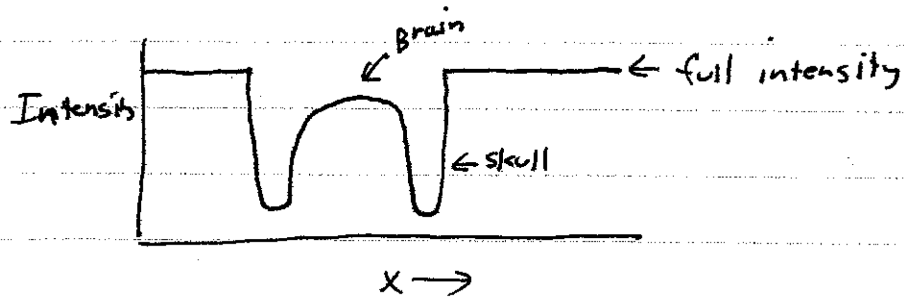
← Stacked slices form 3-D object.

Front view

Now, examine one slice from the top view:



The intensity in the detector, plotted as a function of scan location x, might look like this



If the attenuation per unit length in a tissue of density ρ is proportional to ρ, then the intensity as a function of scan direction x is the integral of the exponential of the density along direction y:

$$i(x) = \int e^{-\rho(x,y)}\, dy$$

if suitable units are chosen for ρ. If ρ is small, the kernel becomes $1 - \rho(x,y)$, otherwise we can operate on the negative logarithm of $i(x)$ and use instead $\rho(x,y)$ directly. We'll assume for now we can somehow express our slice integral as simply

$$i(x) = \int \rho(x,y)\, dy$$

which we recognize as the projection operator along the y direction.

If the object or instrument rotates, we can obtain projections at any angle. How do we relate a series of projections to the original distribution $f(x,y)$?

Circular symmetry review

For circularly symmetric geometries, we found we could express the 2-D Fourier transform as a 1-D function of radius only, according to the Hankel transform

$$F(q) = 2\pi \int_0^\infty f(r)\, J_0(2\pi q r)\, r\, dr$$

We also introduced projection of a circularly symmetric object as the Abel transform:

$$f_A(x) = \int_x^\infty \frac{f(r)}{\sqrt{r^2-x^2}}\, r\,dr$$

or

$$= \int f(x,y)\,dy$$

We also saw that there was a relation between Hankel, Abel, and Fourier transforms as given by

$$\begin{tikzcd}
f(r) \arrow[d, "\text{Abel transform}"'] \arrow[dr, leftrightarrow, "\text{Hankel transform}"] & \\
f_A(x) \arrow[r, leftrightarrow, "\text{Fourier transform}"'] & F(q)
\end{tikzcd}$$

We will now investigate whether any of these relations are meaningful for non-circulary-symmetric distributions. First let's understand the relationship among these more fully for the symmetric distributions.

<u>Abel-Fourier-Hankel Cycle</u>

Let's describe each transform by a script letter to abbreviate the notation. Given a function $f(r)$, we can describe the relation between transforms as

$$\mathcal{H}\mathcal{F}\mathcal{A} f = f$$

That is, the Hankel transform of the Fourier transform of the Abel transform returns the original function. Since this sequence is cyclic, we can summarize the operations as:

$$\mathcal{H}\mathcal{F}A = A\mathcal{H}\mathcal{F} = \mathcal{F}A\mathcal{H} = \mathcal{I}$$

where $\mathcal{I}$ is the identity transformation.

We have looked at examples of these relations before. Let $f(r) = \operatorname{jinc}(r)$. Its Hankel transform is $\operatorname{rect} q$, its Abel transform is $\operatorname{sinc} x$, its Fourier transform $\sqrt{1-(2u)^2}\,\operatorname{rect} u$. These and others are again summarized in the following table.

Table 14-1 Table of Abel-Fourier-Hankel quartets.

f(r) / Abel transform $\mathcal{A}$f	Fourier transform $\mathcal{F}$f / Hankel transform $\mathcal{H}$f
$\operatorname{rect} r$ $\sqrt{1-(2x)^2}\,\operatorname{rect} x$	$\operatorname{sinc} u$ $\operatorname{jinc} q$
$\operatorname{jinc} r$ $\operatorname{sinc} x$	$\sqrt{1-(2u)^2}\,\operatorname{rect} u$ $\operatorname{rect} q$
$\operatorname{jinc}^2 r$ $(4x)^{-2}H_1(4\pi x)$	See Fig. 9-12 $\operatorname{chat} q$
$\operatorname{sinc} r$ $J_0(\pi x)$	$\operatorname{rect} u$ $\pi^{-1}(\frac{1}{4}-q^2)^{-1/2}\operatorname{rect} q$
$\delta(r-a)$ $2a(a^2-x^2)^{-1/2}\Pi(x/2a)$	$2\cos(2\pi a u)$ $2\pi a J_0(2\pi a q)$
$\Lambda(r)$ $(1-x^2)^{1/2} - x^2\cosh^{-1}x^{-1}$	$\operatorname{sinc}^2 u$ $2\pi\left[q^{-3}\int_0^q J_0(x)dx - q^{-2}J_0(q)\right]$
$\exp(-\pi r^2)$ $\exp(-\pi x^2)$	$\exp(-\pi u^2)$ $\exp(-\pi q^2)$

Now, since $\mathcal{H}[\mathcal{H}\mathcal{F}Af] = \mathcal{H}\mathcal{I}f$, since $\mathcal{H}\mathcal{H} = \mathcal{I}$ we also have

$$\mathcal{H}f = \mathcal{F}Af$$

which we used before to evaluate Hankel transforms.

There are several more of these relations:

$$\mathcal{H} = \mathcal{F} A$$
$$A = \mathcal{F} \mathcal{H}$$
$$\mathcal{F} = A \mathcal{H}$$

The simple operator notation, which follows from simple reasoning, replaces the equivalent, complicated integral relations:

$$\int_{-\infty}^{\infty} \mathbf{f}(r) J_0(2\pi q r) 2\pi r \, dr = \int_{-\infty}^{\infty} e^{-i2\pi q x} \left[\int_x^{\infty} \frac{\mathbf{f}(r) r \, dr}{\sqrt{r^2 - x^2}} \right] dx$$
$$\int_x^{\infty} \frac{\mathbf{f}(r) r \, dr}{\sqrt{r^2 - x^2}} = \int_{-\infty}^{\infty} e^{-i2\pi x q} \left[\int_0^{\infty} \mathbf{f}(r) J_0(2\pi q r) 2\pi r dr \right] dq$$
$$\int_{-\infty}^{\infty} e^{-i2\pi x r} \mathbf{f}(r) dr = \int_x^{\infty} \frac{r}{\sqrt{r^2 - x^2}} \left[\int_0^{\infty} \mathbf{f}(r) J_0(2\pi r q) 2\pi q \, dq \right] dr.$$

Inverse cycle

Using inverse transforms, there are three inverse cyclic relations:

$$\mathcal{F} \mathcal{H} A^{-1} = \mathcal{H} A^{-1} \mathcal{F} = A^{-1} \mathcal{F} \mathcal{H} = \mathcal{I}$$

and three corresponding substitutions:

$$\mathcal{H} = A^{-1} \mathcal{F}$$
$$A^{-1} = \mathcal{H} \mathcal{F}$$
$$\mathcal{F} = \mathcal{H} A^{-1}$$

This yields an integral for the inverse Abel transform:

$$f(r) = A^{-1} f_A(x) = \int_0^{\infty} J_0(2\pi r q) \left[\int_{-\infty}^{\infty} f_A(x) e^{-i2\pi q x} dx \right] 2\pi q \, dq$$

This form may be more easily soluble than the inverse Abel definition we encountered previously.

Four-stage cycle

Finally, by substituting the two-operator relations into the three-operator relations, we obtain 4-cycle relations:

$$\mathcal{F} A \mathcal{F} A = A \mathcal{F} A \mathcal{F} = \mathcal{I}$$

or

$$\mathcal{F} A^{-1} \mathcal{F} A^{-1} = A^{-1} \mathcal{F} A^{-1} \mathcal{F} = \mathcal{I}$$

hence

$$A = \mathcal{F} A^{-1} \mathcal{F} \quad \text{and} \quad A^{-1} = \mathcal{F} A \mathcal{F}$$

so now we have alternate methods of evaluating Abel and inverse Abel transforms.

Projection-slice theorem

Now we can consider non-symmetric distributions. The projection-slice theorem is the basis of many reconstruction devices.

Start with

$$\mathcal{H} f(r) = \mathcal{F}(A f(r))$$

or the Hankel transform can be arrived at by the Fourier transform of the Abel transform, or projection.

In other words, "the projection" $A f(r)$ and the "slice" $F(u,0)$ through the 2-D Fourier transform $F(u,v)$ are a one-dimensional Fourier transform pair. Substituting the 2-D Fourier transform for the Hankel transform removes the requirement for circular symmetry.

Thus, we can state the Projection-Slice Theorem:

"The projection of $f(x,y)$ in direction θ is the 1-D Fourier transform of the slice through $F(u,v)$ in the same direction"

<u>Proof:</u> Consider the 2-D Fourier transform relation

$$F(u,v) = \iint_{-\infty}^{\infty} f(x,y)\, e^{-i 2\pi (ux+vy)}\, dx\, dy$$

Let $v=0$ to obtain the slice along u:

$$F(u,0) = \int_{-\infty}^{\infty} \underbrace{\left[\int_{-\infty}^{\infty} f(x,y)\, dy \right]}_{\text{projection along } \theta = 0} e^{-i 2\pi ux}\, dx$$

Hence the slice of $F(u,v)$ along $v=0$ is simply the one-D transform of the projection at $\theta = 0$, or along y.

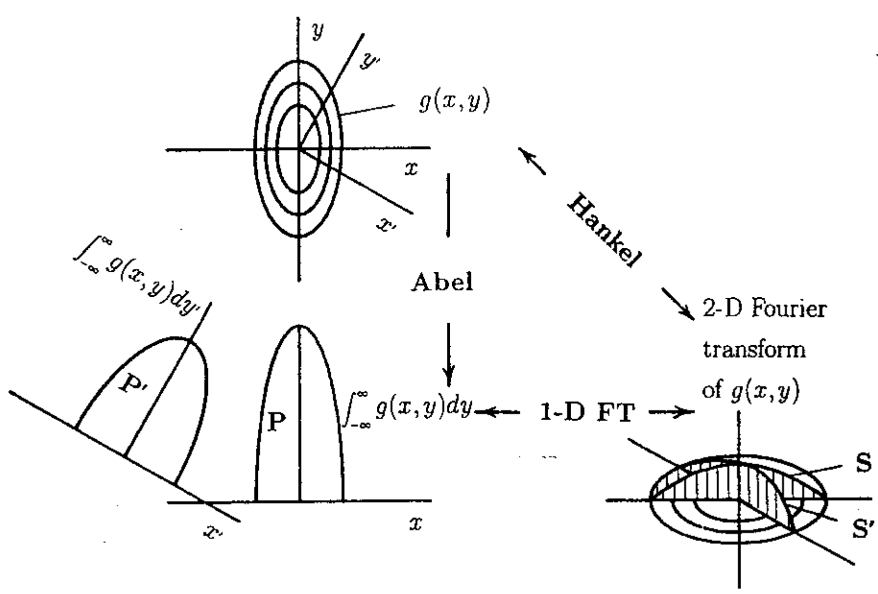


Figure 14-5 Synoptic chart of the relationship of line integration to one- and two-dimensional Fourier transforms (see Bracewell, 1956).

In terms of the projection operator P_Θ we described early in the quarter, we can write the previous equation as

$$ {}^1\mathcal{F}\{P_0 f(x,y)\} = \left[{}^2\mathcal{F} f(x,y)\right]_{\theta=0} $$

How about other angles? Simply invoke the rotation theorem, which states that the transform of a rotated distribution is the original Fourier transform suitably rotated. Then the above generalizes to:

$$ {}^1\mathcal{F}\{P_\theta f(x,y)\} = \left[{}^2\mathcal{F} f(x,y)\right]_{\theta=\theta'} $$

Therefore, we can reconstruct a function from its slices by suitably transforming them and building up the 2-D transform, which we invert to obtain $f(x,y)$.