Restoration in the presence of noise

We have seen that imaging systems typically exhibit nonuniform response to spatial frequency, and there may even be gaps in the spectrum that are not measured at all. But assuming that the transfer function is known, the signal may be reconstructed except where there is no sensitivity to the spectrum.

This reconstruction had the form of devising a "filter" $F(s)$ that could compensate for the system transfer function $H(s)$, where

$$F(s) = \frac{1}{H(s)}$$

which can be evaluated anywhere $H(s) \neq 0$.

This simple analysis ignored a basic fact of the real world - signals are recorded with "noise", that is the measured values do not exactly mirror the "true" values.

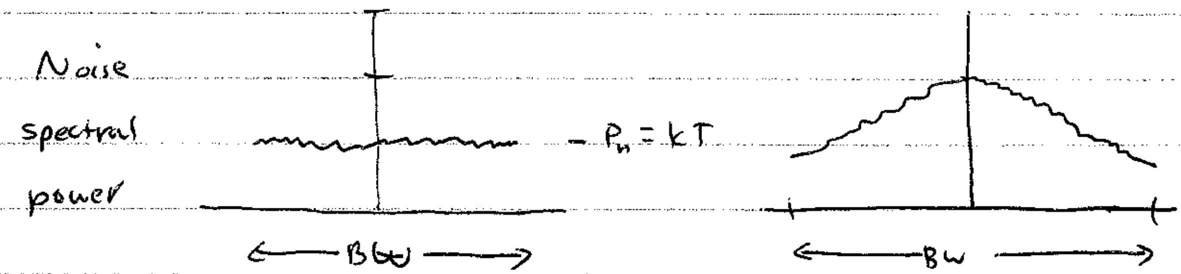
Noise

Noise is generated by many processes, some natural and beyond our control, and some due to instrumental limitations.

Natural sources: "thermal" noise in detectors, ~~light~~ scattering in the atmosphere, "speckle" noise in coherent imaging, quantum effects, natural background radiation

Instrumental sources: quantization, sidelobes, stray light in an optical system, nonlinearities, man-made interference

These noises themselves may have spectral shapes, or they may be "white". The most common model for noise is the white Gaussian process, spectrally uniform and uncorrelated with space or time.



White noise "Pink" noise

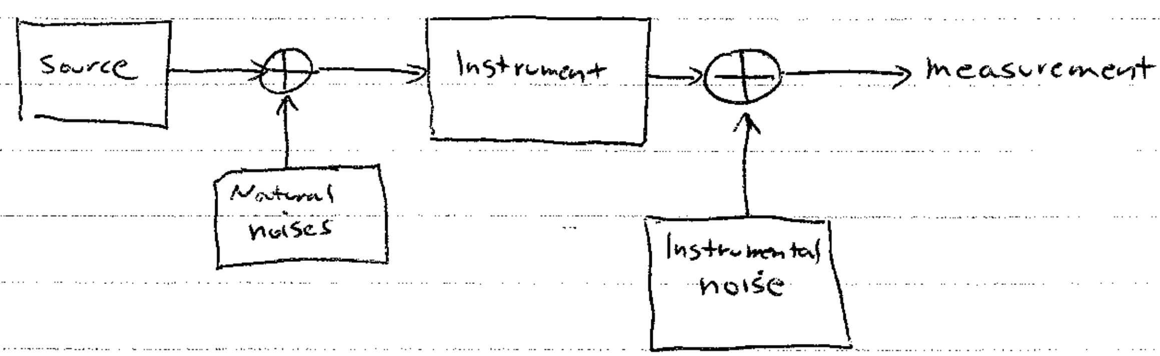
Let's look at white processes here. This process is modeled in terms of a noise temperature T, and the total power associated with noise P_n within a bandwidth BW is

$$P_n = kT\,BW$$

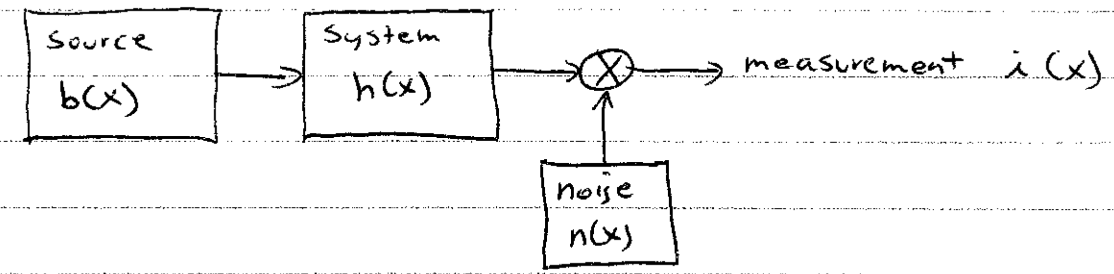
where P_n is the noise in watts, T is the noise temperature in degrees, BW is the bandwidth, and k is Boltzmann's constant.

Modeling of systems with noise

We could model a realistic system as follows:



It is typically to lump all noises together into one "equivalent noise" source, which is a good model if all sources are white Gaussian. Then we have



Our model of this process becomes

$$i(x) = h(x) * b(x) + n(x)$$

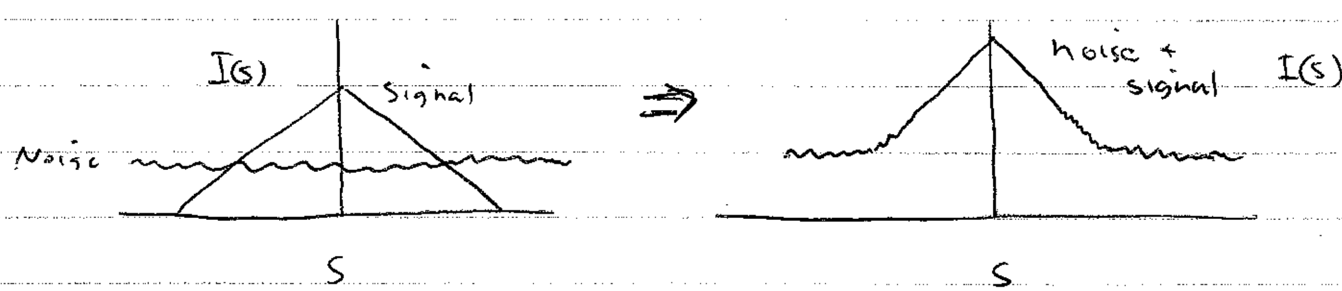
or

$$I(s) = H(s)\, B(s) + N(s)$$

Clearly this generalizes to two dimensions.

Spectrum of noisy measurement

If we were to plot the spectrum of our noisy signal we'd see



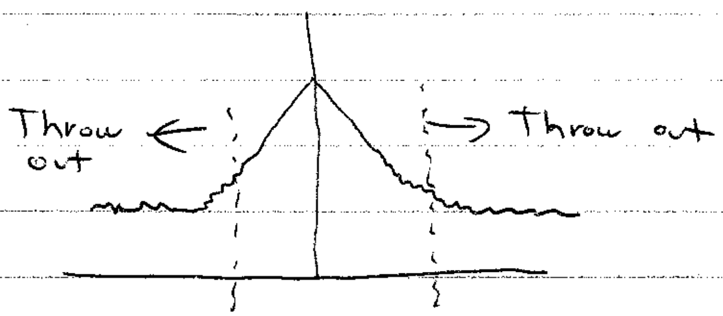
where the triangular spectrum represents the product of the true brightness $B(s)$ and the transfer function $H(s)$

What does this imply for our reconstruction?

Note that in some regions of the spectrum, the noise power exceeds the signal, and in others the two are comparable. Finally, in some cases the signal dominates.

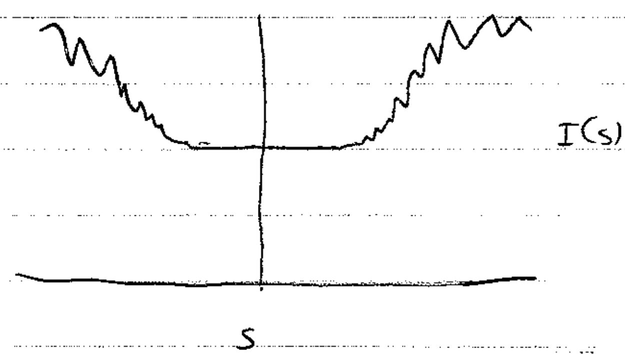
How can we obtain the "best" image?

- Truncate spectrum, saving only parts where signal dominates



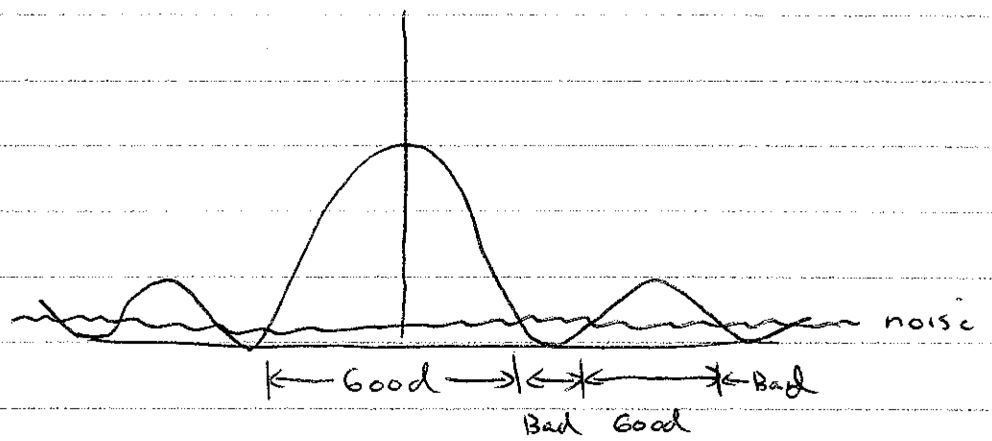
Eliminating high frequencies gets rid of regions where noise dominates. This is again a low-pass filter. But what does this do to our reconstruction?

Recall $F(s) = \frac{1}{H(s)}$. Applying this filter to the noisy signal results in a spectrum



Note here that most of the power is in the very noisy region. Low-pass filtering may be the only thing we can do.

Suppose our transfer function was a sinc^2 pattern in power. (Uniform aperture). The situation:



Again, some regions are going to be dominated by signal, some by noise. In this case simple lowpass filtering will needlessly remove good data in the spectrum.

<u>More rigorous procedure</u>

Consider our model

$$i(x) = h(x) * b(x) + n(x)$$

We'll look for a filter $f(x)$ to do the best job of estimating our signal. If a filter $f(x)$ leads to an estimate of $b(x)$ denoted $b_{est}(x)$, we can say the error $e(x)$ is defined as

$$e(x) = b_{est}(x) - b(x)$$

and say the optimal filter minimizes (in the least-square sense)

$$\int_{-\infty}^{\infty} |e(x)|^2 dx$$

By Rayleigh's theorem, it is the same as minimizing

$$\int_{-\infty}^{\infty} |E(s)|^2 ds$$

Since

$$b_{est}(x) = f(x) * [h(x) * b(x) + n(x)]$$

$$e(x) = f(x) * [h(x) * b(x) + n(x)] - b(x)$$

and

$$E(s) = F(s)[H(s)B(s) + N(s)] - B(s)$$

We want to determine $e(x)$ (or $E(s)$) to minimize our error integral.

<u>Ideal restoration case</u>

If $h(x)$ is real and even, $H(s)$ is also real. In this case the algebra is simplified. If $\langle\cdot\rangle$ denotes averaging over ensembles,

$$\langle |E|^2 \rangle = \langle [F(HB+N) - B][F(HB^*+N^*) - B^*] \rangle$$

$$= F^2 \langle (HB+N)(HB^*+N^*) \rangle - F \langle B^*(HB+N) + B(HB^*+N^*) \rangle - \langle BB^* \rangle$$

where F is assumed real, which is a consequence of $h(x)$ being real and even.

This has the form

$$\langle |E|^2 \rangle = \alpha F^2 + \beta F + \gamma$$

Differentiating, equating to zero then yields the optimum at $\frac{-\beta}{2\alpha}$

or

$$F_{opt} = \frac{2\langle HB^*B \rangle + \langle B^*N \rangle + \langle BN^* \rangle}{2[\langle HBHB^* \rangle + \langle HB^*N \rangle + \langle HBN^* \rangle + \langle NN^* \rangle]}$$

Now, define $G(s) = B(s)\,H(s)$ as the spectrum of the measurement in the noise-free case. Substituting back into the above

$$F_{opt} = \frac{1}{H} \cdot \frac{\langle GG^* \rangle + \frac{1}{2}\langle G^*N \rangle + \frac{1}{2}\langle GN^* \rangle}{\langle GG^* \rangle + \langle G^*N \rangle + \langle GN^* \rangle + \langle NN^* \rangle} \qquad s < \text{cutoff value}$$

$$= 0 \qquad s > \text{cutoff value}$$

where the cutoff value is chosen as a tradeoff between the highest frequency to be retained and the amount of noise to admit in the solution.

We usually denote $\langle NN^* \rangle$ as the noise power spectrum and $\langle GG^* \rangle$ as the "signal" power spectrum, where signal here means what we observe with our instrument in the absense of noise, not the desired brightness $B(s)$.

If the noise is zero-mean and uncorrelated with the signal, the cross-spectra $\langle G^*N \rangle$ and $\langle GN^* \rangle$ are zero. In this case the solution F_{opt} reduces to the simple form

$$F_{opt} = \frac{1}{H} \; \frac{\langle GG^* \rangle}{\langle GG^* \rangle + \langle NN^* \rangle}$$

$$= \frac{1}{H} \; \frac{\text{signal power}}{\text{signal power} + \text{noise power}}$$

$$= \frac{1}{H} \frac{P_s}{P_s + P_N}$$

$$= \frac{1}{H} \; \frac{P_s/P_N}{P_s/P_N + 1}$$

The ratio P_S/P_N is often called signal to noise ratio, or SNR. Thus the filter is expressible

$$F_{opt} = \frac{1}{H} \frac{SNR}{SNR+1} \qquad s < \text{cutoff}$$

$$= 0 \qquad s > \text{cutoff}$$

SNR is a function of spectral frequency s. For high SNR region, the result approaches $\frac{1}{H}$, as is the ideal case without noise. As $SNR \to 0$, the filter also goes to zero, limiting sensitivity to the portion of the spectrum that is most corrupted by noise.

Some special cases:

1. No noise $F_{opt} = \frac{1}{H(s)}$

2. Uncorrelated, zero mean noise: $F_{opt}(s) = \frac{1}{H(s)} \frac{|G(s)|^2}{|G(s)|^2 + |N(s)|^2}$

3. Noise proportional to signal (multiplicative noise, coherent noise:)

We have $\langle |N|^2 \rangle = K \langle |G|^2 \rangle$ and $\langle G N^* \rangle = 0 = \langle G^* N \rangle$

$$F_{opt}(s) = \frac{1}{H(s)(1+K)}$$

4. $H(s) = \Lambda(s)$ Triangular transfer function

$B(s) = 1$

$\langle |N(s)|^2 \rangle = K$

$\langle GN^* \rangle = \langle G^* N \rangle = 0$

$$F_{opt}(s) = \frac{1}{\Lambda(s)} \cdot \underbrace{\frac{\Lambda^2(s)}{\Lambda^2(s) + K}}$$

↑ modification to ideal filter $\frac{1}{\Lambda(s)}$

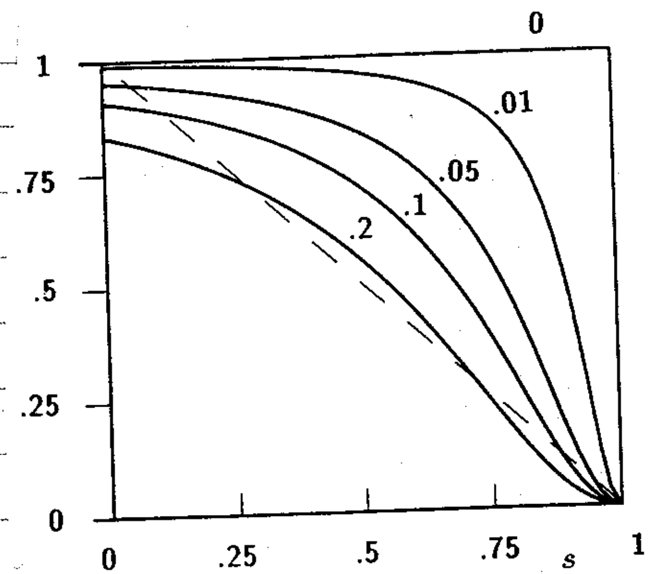


Figure 13-15 The modifying factor $\Lambda^2/(\Lambda^2 + K)$ for various strengths of noise power K with reference to a triangular transfer function $\Lambda(s)$ (broken line).

For thought – compare how restoration techniques apply to several different aperture functions:

1. $f(x,y) = {}^2\mathrm{rect}(x,y)$
2. $f(x,y) = {}^2\mathrm{sinc}(x,y)$
3. $f(x,y) = \cos(\pi x)\cos(\pi y)\,{}^2\mathrm{rect}(x,y)$
4. $f(x,y) = \mathrm{rect}(r)$
5. $f(x,y) = \mathrm{jinc}(r)$