(Much of this material is found in the text, Chapter 13)

Restoration

We have seen how it is possible to populate the u-v plane with visibility islands and thereby reconstruct, through synthesis, the Fourier spectra of signals. How does this actually work, and what are the limitations to the procedure?

To begin with, we will consider the more limited problem of correcting a measurement for known instrumental effects, which Bracewell calls restoration. Briefly stated, the problem is this: Since many instruments record a filtered version of the original spectrum, how might we "correct" the signal to obtain more of the original spectrum's relative weights at different spatial frequencies?

Transfer function viewpoint

Our convolutional model of imaging is (1-D case)

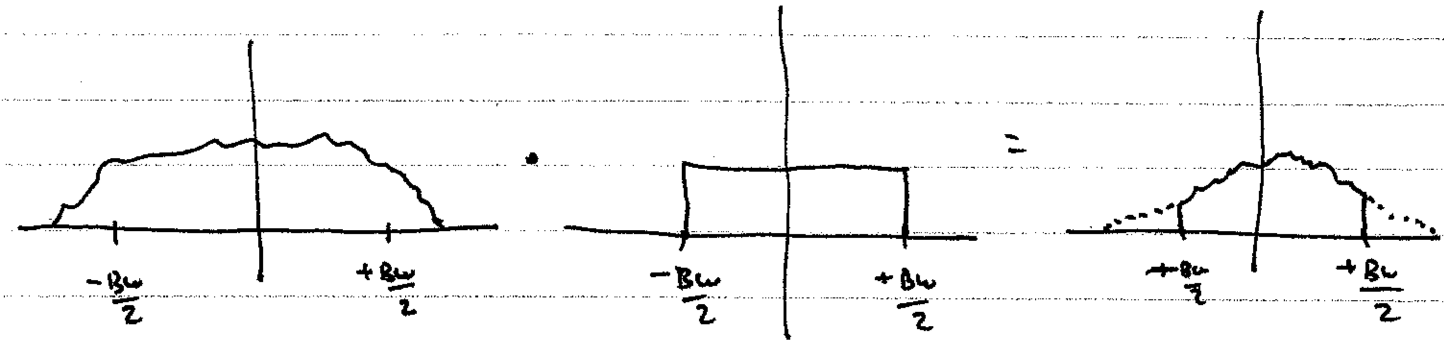
$$i(x) = h(x) * b(x)$$

where the image $i(x)$ is the convolution of the impulse response $h(x)$ with the true brightness distribution $b(x)$.

In the transform domain,

$$I(s) = H(s) \cdot B(s)$$

Pictorially, consider the case of a limited bandwidth, equal amplitude filter defined by $H(s) = \text{rect}\,\frac{s}{BW}$ where BW is the bandwidth of the filter:



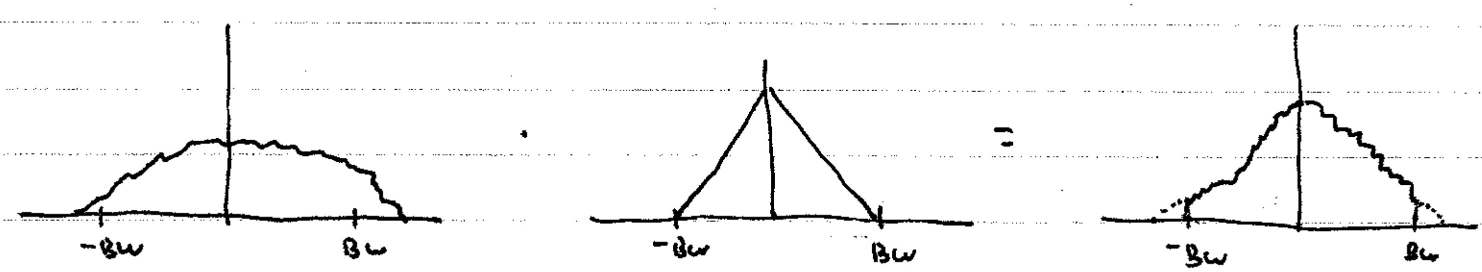
B(s) H(s) I(s)

The image spectrum I(s) faithfully reproduces the brightness source B(s) over the region from $-Bw/2$ to $Bw/2$, but beyond that the response is zero. The dotted sections of the spectrum are lost.

Can the dotted portion of the spectrum be recovered from the image I(s) or $i(x)$?

This example is an extreme case where irrecoverable damage is done by the instrument, as represented by its transfer function.

Now, consider the case where the ~~transfer~~ impulse response is $\text{sinc}^2(X \cdot Bw) \cdot Bw$. Then the transfer function will be $H(s) = \Lambda\left(\frac{s}{Bw}\right)$. The situation here is different:



B(s) H(s) I(s)

Here some signal beyond the bandpass is lost, but there is a weighting applied to the spectrum.

Now, not only is some signal lost, but the passband is distorted. How might we compensate for this further distortion?

Suppose we define a "restoration filter" $R(s)$ according to

$$R(s) = \frac{1}{\Lambda\left(\frac{s}{Bw}\right)} \cdot \mathrm{rect}\left(\frac{s}{2Bw}\right)$$

We could multiply the image $I(s)$ by this filter to obtain a restored image $I_{restored}(s)$:

$$I_{restored}(s) = I(s) \cdot R(s) = H(s) \cdot R(s) \cdot B(s) = B(s)\ \mathrm{rect}\left(\frac{s}{2Bw}\right)$$

The restored image then has the original spectrum values, save for the portions lost at the edges. If we designed the instrument with sufficient resolution to begin with, we can in principle recover the signal completely.

We shall see later that this approach, though promising, is limited in principle by the presence of "noise" in the system. In the meantime, let's examine some algorithms for restoring images.

<u>Successive substitutions</u>

Begin with our convolutional model

$$i(x) = h(x) * b(x)$$

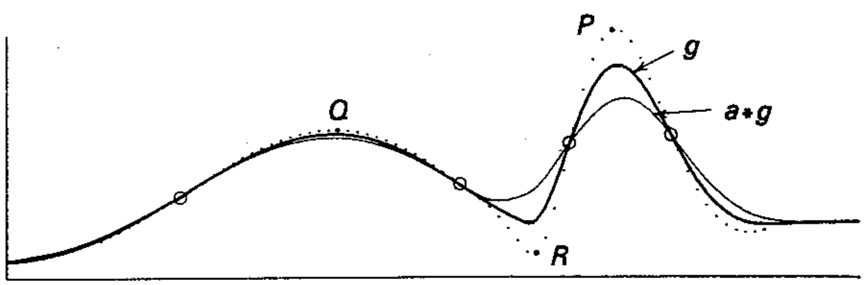
and suppose further that $\int_{-\infty}^{\infty} h(x)\,dx = 1$. (what does this imply for $H(0)$?)

Now, suppose the image $i(x)$ is imaged once again with the same system, then the new image will be further smoothed by $h(x)$ to obtain

$$\text{new image} = h(x) * i(x)$$

and the change in the image would be

$$\text{change} = h(x) * i(x) - i(x)$$



← different notation!

Figure 13-1 A given function g, which is derived from an unknown original f by smoothing, changes to $a * g$ (thin line) if subjected to further smoothing. Reversing the changes leads to a first approximation (dotted line) to the original unsmoothed function.

The application of $h(x)$ to the image further depresses peaks and fills in troughs, but has little effect on the smoother portions of the curve. (How might we understand this from the spectral viewpoint?)

Reasoning that the same thing must have happened during the imaging process, we argue we can obtain a first-order approximation to $b(x)$, $b_1(x)$, by subtracting the change from the image $i(x)$:

$$b_1(x) = i(x) - (h(x) * i(x) - i(x))$$

Thus $b_1(x)$ is expected to be closer to $b(x)$ than was $i(x)$.

How might we test this solution? We can compute the forward relation and compare it with the original measurement:

$$\text{compare } i(x) \text{ with } h(x) * b_1(x)$$

If the two are equal, we have restored $b_1(x)$ within the limits of the instrument. If not, we could compute a second order approximation

$$b_2(x) = b_1(x) - (h(x) * b_1(x) - i(x))$$

and compare as above. The general expansion would be

$$b_n(x) = b_{n-1}(x) - (h(x) * b_{n-1}(x) - i(x))$$

So, does such a sequence converge? We can determine that by examining the algorithm in transform space. Start with

$$I(s) = H(s) \cdot B(s)$$

and

$$\frac{I(s)}{H(s)} = \frac{I(s)}{1-[1-H(s)]}$$

$$= I(s) \cdot \left[1 + (1-H(s)) + [1-H(s)]^2 + \cdots\right]$$

$$= I(s) + [1-H(s)]\, I(s) + [1-H(s)]^2\, I(s) + \cdots$$

Take the first two terms of the sequence and retransform, obtaining

$$i(x) + [\delta(x) - h(x)] * i(x) = i(x) + [i(x) - h(x) * i(x)]$$

which is our first approximation $b_1(x)$. A little algebra shows that the additional terms yield $b_2(x)$, $b_3(x)$, etc.

In this case we know how to evaluate convergence, and the criterion is

$$|1 - H(s)| < 1$$

so that convergence depends on the transfer function $H(s)$. In our previous case where $h(x) = \operatorname{sinc}^2 x$,

$$H(s) = \Lambda(s)\, \operatorname{rect}\left(\frac{s}{2}\right)$$

$$= (1 - |s|)\operatorname{rect}\frac{s}{2}$$

Here $(1 - |s|)$ satisfies convergence except where $|s| \geq 1$, where we already know $I(s)$ is zero. Hence for $\operatorname{sinc}^2()$ instrumental responses we would obtain convergence.

What does the sequence converge to in this case? Since

$$B_1(s) = I(s) + (1 - H(s))\, I(s)$$

$$= I(s) + |s|\, I(s)$$

for $|s| < 1$. Also we know

$$I(s) = H(s)\, B(s)$$
$$= (1 - |s|)\, B(s)$$

so that

$$B_1(s) = (1 + |s|)(1 - |s|)\, B(s)$$
$$= (1 - |s|^2)\, B(s)$$

and continuing on to higher order terms, we see

$$B_n(s) = (1 - |s|^{n+1}) B(s)$$

Plotting successive versions of the effective transfer function $\frac{B_n(s)}{B(s)}$:

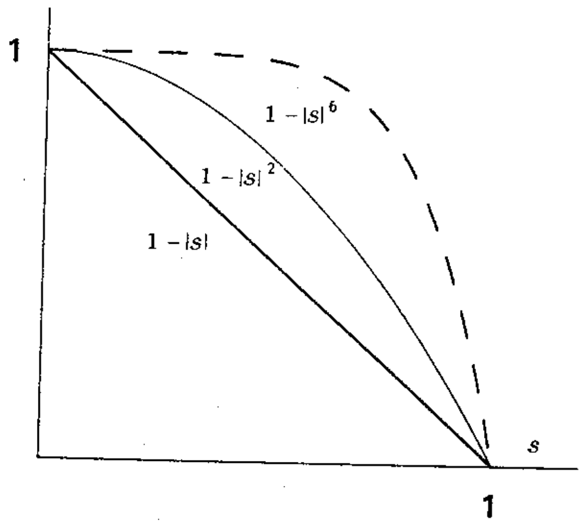


Figure 13-3 Effective transfer functions after one (thin line) and five (broken line) stages of restoration. The limiting effective transfer function is unity where $-1 < s < 1$ and zero elsewhere, perfection for the low frequencies and rejection for the high.

In the limit $1 - |s|^n \rightarrow \text{rect}\,\frac{s}{2}$, and the signal is restored completely.

Another example - running averages

In many cases we scan an object with a uniform aperture, for example, the sound portion of a movie is scanned by a slit aperture. The A/D converter we described before also had the property of averaging over an interval. Both of these are modeled by the "running mean" or "running average" - the mean value of a process over an interval is recorded.

Running means also show up whenever we smooth noisy data before analysis - meteorology is full of these examples.

The convolution kernel for all of these is the rect ( ) function, and we have for our model

$$i(x) = \text{rect}\, x * b(x)$$

in the time (spatial) domain, and in the frequency domain

$$I(s) = \text{sinc}\, s \cdot B(s)$$

What is the convergence criterion for the successive approximation algorithm? Plot $1 - \text{sinc}\, s$:

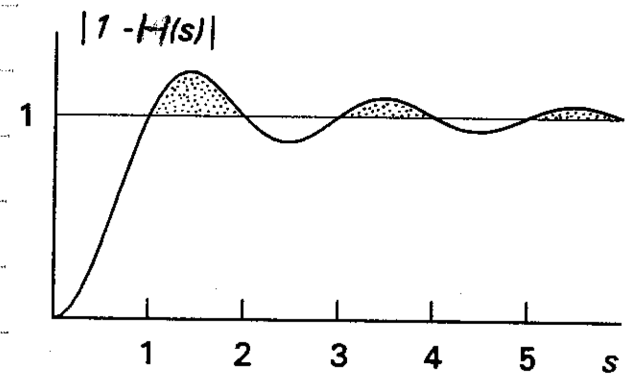


Figure 13-4 Convergence condition is not met in the stippled zones.

Cleary if B(s) has any energy lying in the zones of non-convergence the algorithm won't converge. However, for some parts of the spectrum we might have convergent behavior, so applying a few of the series of successive approximations might generate a more desirable image.

<u>Invisible distributions</u>

Note that H(s) in the above has zeroes at $n = S$. Therefore any distribution, or components of any distribution, compose distributions that would not be imaged by the instrument. These represent "dead zones" in the instrumental response. In this particular case all periodic functions of unit period are invisible.

Eddington's formula

The first historical reference to restoration is a correction formula due to Eddington, derived to increase spectral resolution in optical spectrometers. Eddington derived the following, which is valid if the instrumental response is Gaussian:

$$b(x) = i(x) - \frac{1}{4} i''(x) + \frac{1}{32} i''''(x) - \cdots$$

If there are a series of elements in an optical instrument, the total transfer function is given by multiple convolutions of each element. By the central limit theorem we know we can model the composite transfer function by a Gaussian shape, so in many practical cases this is a good approximation.

We can derive the formula as follows: let

$$h(x) = w^{-1} e^{-\pi x^2/w^2}$$

and

$$H(s) = e^{-\pi w^2 s^2}$$

As before,

$$I(s) = H(s)\ B(s)$$

$$B(s) = I(s)/H(s)$$

$$= I(s)\left[1 + \pi w^2 s^2 + (\pi w^2 s^2)^2 \cdot \frac{1}{2} + \cdots \right]$$

Now, the derivative theorem states

$$f'(x) \supset i 2\pi s\, F(s)$$
$$f''(x) \supset (i 2\pi s)^2 F(s)$$

and so forth. Substituting into the above

$$B(s) = I(s) - \frac{w^2}{4\pi}(i2\pi s)^2 I(s) + \frac{w^4}{32\pi^2}(i 2\pi s)^4 I(s) - \cdots$$

or

$$b(x) = i(x) - \frac{w^2}{4\pi} i''(x) + \frac{w^4}{32\pi^2} i''''(x) - \cdots$$

We would have obtain Eddington's result exactly if $h(x) = \pi^{-1/2} e^{-x^2}$
This formula has been used extensively, but in practice only the first term is used because of noise considerations when calculating high-order derivatives from ~~samp~~ measurements.