More on radio astronomy and interferometers.

We have shown the relationship between the image spectrum and the original brightness spectrum in terms of the autocorrelation of the illumination aperture:

$$I(s) = f(x) \star f(x) \cdot B(s)$$

and shown the extension to the adding interferometer. It is obvious also how the field patterns are related to the apertures, such as:

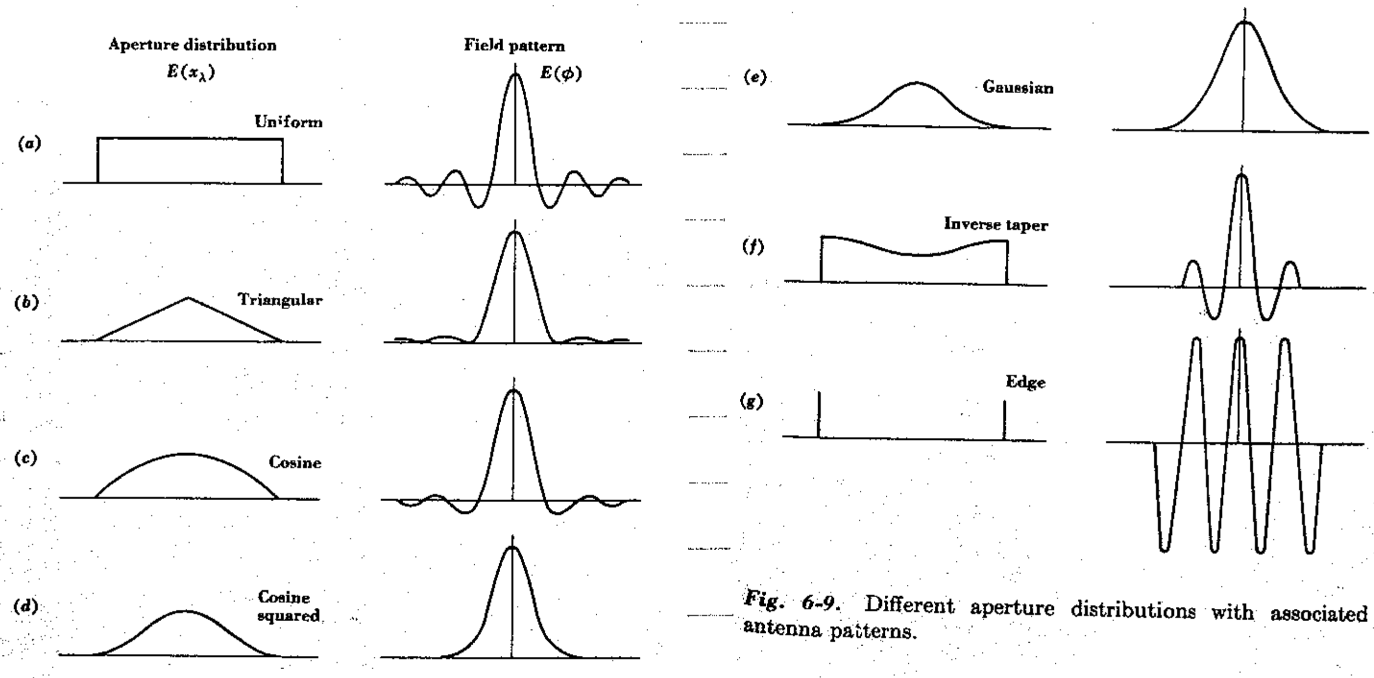


Fig. 6-9. Different aperture distributions with associated antenna patterns.

The intensity patterns are the squares of the field patterns, the convolution kernels are the autocorrelations of the apertures.

Recall for the adding interferometer, the autocorrelation function was

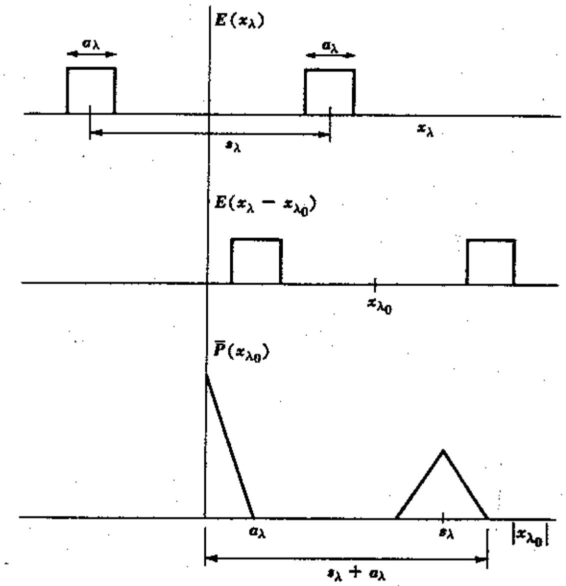


Fig. 6-13. **Autocorrelation function of aperture distribution of simple interferometer.**

The point source array pattern follows:

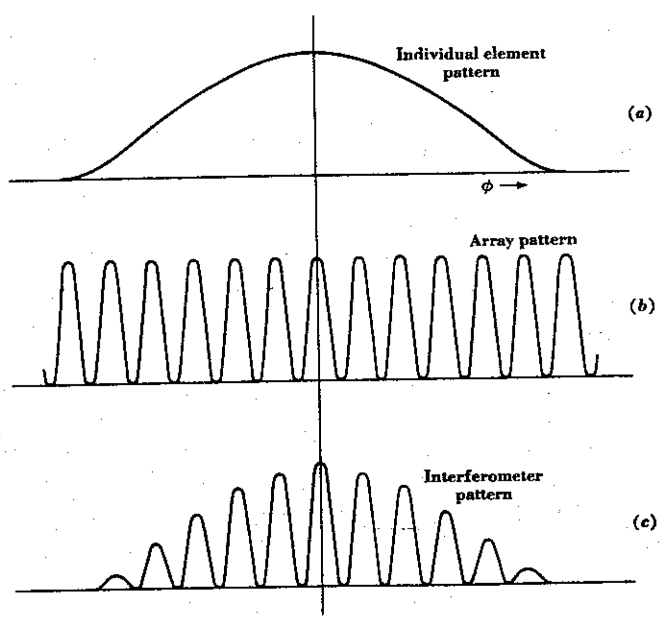


Fig. 6-14. **(*a*) Individual-element pattern; (*b*) array pattern; and (*c*) the resultant interferometer pattern for the case of a point source.**

For a source that is spread out in the sky, the patterns of received power might be:

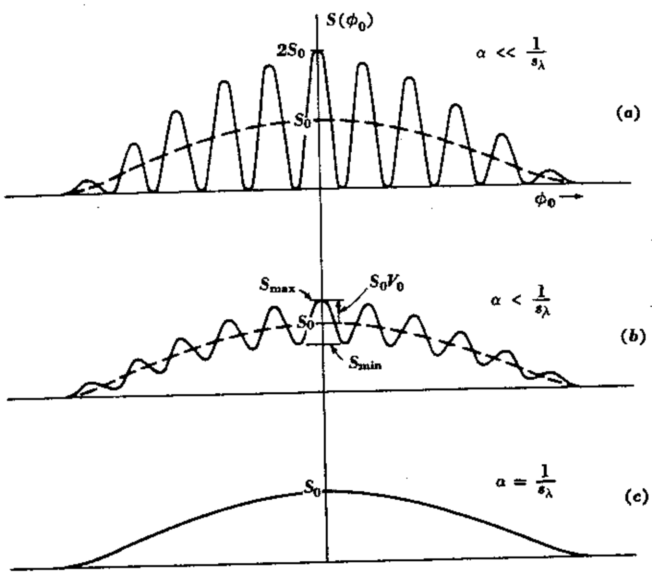


Fig. 6-15. Interferometer pattern (*a*) for point source; (*b*) for a uniform extended source of angle $\alpha < 1/s_\lambda$; and (*c*) for a uniform extended source of angle $\alpha = 1/s_\lambda$.

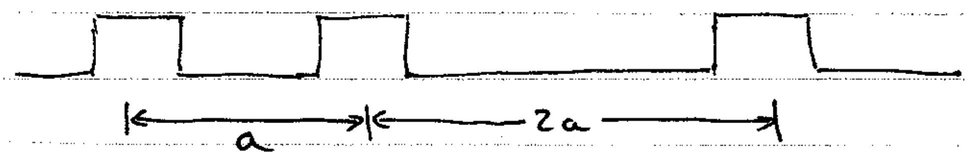
The extent of the response is driven by the element pattern, while the frequency of the "ripples" is determined by the element spacing. The observed pattern, remember, is the convolution of the actual pattern with the intensity pattern of the telescope:

Forming an image

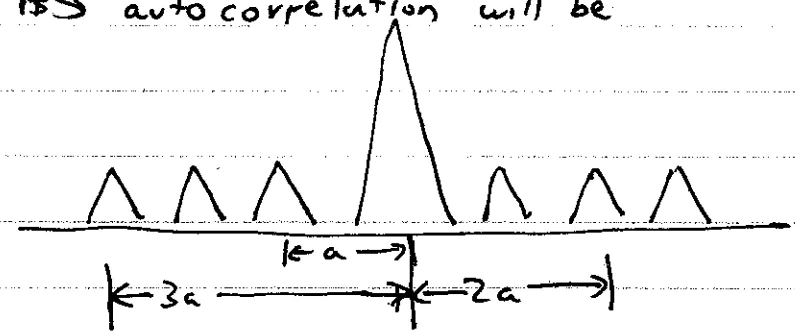
We have seen that the measured spectrum $I(s)$ is related to the true sky brightness $B(s)$ via the autocorrelation of the aperture function:

$$I(s) = f(x) \star f(x) \cdot B(s)$$

The more "complete" that the transfer function $f(x) \star f(x)$ is, the more accurate our image will be. Consider an aperture of the form



Its autocorrelation will be

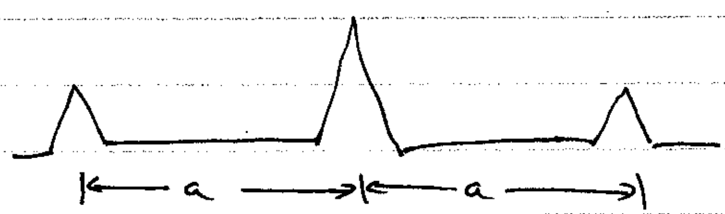


This is much more complete than our simple 2 element interferometer. Note that the central peak is 3 times as high as the other peaks, so that we would weight this part of the spectrum by $1/3$ before computing the inverse transform to recover the image.

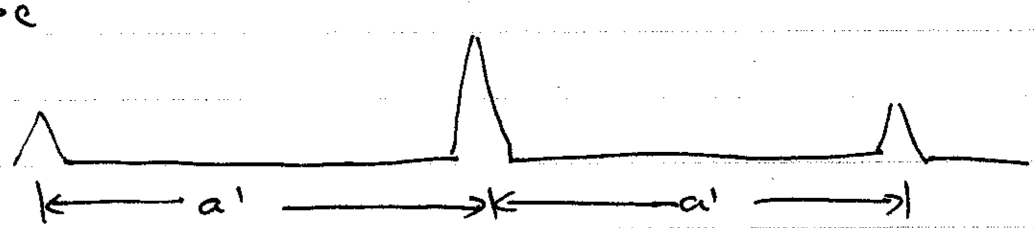
For a more complex aperture, we can easily compute the autocorrelation, which gives us the transfer function. This can be weighted appropriately to obtain a spectrum as flat as possible, so that the impulse response of the system approaches a $\delta(\theta)$.

2-element interferometer with variable spacing

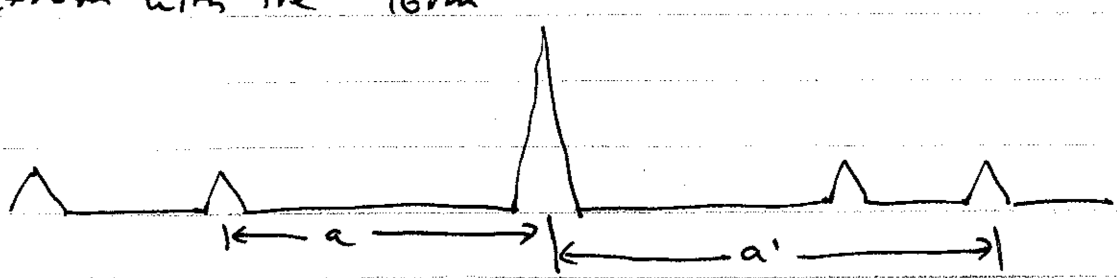
Another approach is to use a single pair of antennas at variable spacing to synthesize a more complete aperture. Recall that for a single pair of antennas, the autocorrelation is



where a is the element spacing expressed in frequency. Suppose we can vary this spacing, to a'. Then the transfer function might be



We can add the spectrum measured at spacing a, $I(a)$, to that measured at a', $I(a')$ to obtain a composite spectrum with the form

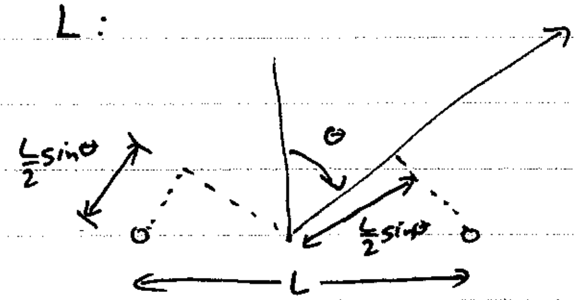


With enough spacings we can sample the full spectrum of the sky and reconstruct the image. We need to account for the multiple measurements of the zero-frequency part of the spectrum, but in essence we measure the Fourier transform of the sky one component at a time.

Correlating Interferometer

The adding interferometer permits the synthesis of large arrays but has the problem of over-measuring the near-zero part of the spectrum. It is also a technical challenge sum the signals from many antennas properly. Let's examine another type of interferometer, the correlating interferometer, in which we measure the pairwise product of signals from antenna elements, and then reconstruct the image.

As before, start with a pair of antennas separated by a distance L:



We receive two signals at the antennas, one advaced in phase by $\frac{2\pi}{\lambda}\frac{L}{2}\sin\theta$, and one retarded by the same amount when each is referenced to the center of the array.

Suppose that the sky can be characterized by an amplitude signal $a(\theta)$, the $|\text{square}|$ of which is the desired brightness $b(\theta)$. Suppose further that each antenna has an element pattern $e(\theta)$. Then we can express the signal measured at antenna 1 as an integration over the element pattern with the antenna pointed at some angle θ_0 as

$$s_1(\theta_0) = \int a(\theta)\, e(\theta-\theta_0)\, e^{-i \frac{2\pi}{\lambda} \frac{L}{2} \sin\theta}\, d\theta$$

Similarly, at antenna 2

$$s_2(\theta_0) = \int a(\theta)\, e(\theta-\theta_0)\, e^{+i \frac{2\pi}{\lambda} \frac{L}{2} \sin\theta}\, d\theta$$

Since the antenna limits the response to small $\theta-\theta_0$, let $x = \theta - \theta_0$, $\theta = x+\theta_0$, and $dx = d\theta$. Then

$$s_1(\theta_0) = \int a(x+\theta_0)\, e(x)\, e^{-i \frac{2\pi}{\lambda} \frac{L}{2} \sin(x+\theta_0)}\, dx$$

Since x is small,

$$\sin(x+\theta_0) = \sin x \cos\theta_0 + \cos x \sin\theta_0$$

$$\approx x \cos\theta_0 + \sin\theta_0$$

and

$$e^{-i \frac{2\pi}{\lambda} \frac{L}{2} \sin(\theta_0+x)} = e^{-i \frac{2\pi}{\lambda} \frac{L}{2} x \cos\theta_0}\, e^{-i \frac{2\pi}{\lambda} \frac{L}{2} \sin\theta_0}$$

Then

$$s_1(\theta_0) = e^{-i \frac{2\pi}{\lambda} \frac{L}{2} \sin\theta_0} \int a(x+\theta_0)\, e(x)\, e^{-i \frac{2\pi}{\lambda} \frac{L}{2} x \cos\theta_0}\, dx$$

and

$$s_2(\theta_0) = e^{+i \frac{2\pi}{\lambda} \frac{L}{2} \sin\theta_0} \int a(x+\theta_0)\, e(x)\, e^{+i \frac{2\pi}{\lambda} \frac{L}{2} x \cos\theta_0}\, dx$$

Instead of adding these two signals, form their product after conjugating $s_2(\theta_0)$:

$$s_1(\theta_0)\, s_2^*(\theta_0) = e^{-i\frac{2\pi}{\lambda}\frac{L}{2}\sin\theta_0}\, e^{-i\frac{2\pi}{\lambda}\frac{L}{2}\sin\theta_0} \cdot$$

$$\iint a(x+\theta_0)\, a^*(x'+\theta_0)\, e(x)\, e^*(x')\, e^{-i\frac{2\pi}{\lambda}\frac{L}{2}x\cos\theta_0} \cdot$$

$$e^{-i\frac{2\pi}{\lambda}\frac{L}{2}x'\cos\theta_0}\, dx\, dx'$$

Define the complex visibility $V(\theta_0) = s_1(\theta)\, s_2^*(\theta_0)\, e^{i2\pi\frac{L}{\lambda}\sin\theta_0}$, and thus

$$V(\theta_0) = \iint a(x+\theta_0)\, a^*(x'+\theta_0)\, e(x)\, e^*(x')\, e^{-i\frac{2\pi}{\lambda}\frac{L}{2}\cos\theta_0(x+x')}\, dx\, dx'$$

If $a(\theta)$ is uncorrelated over space, as we would expect from a collection of naturally-occurring black bodies, then

$$\langle a(x+\theta_0)\, a^*(x'+\theta)\rangle = \langle b(x+\theta_0)\, \delta(x-x')\rangle$$

so we can write $V(\theta_0)$ as

$$V(\theta_0) = \iint b(x+\theta_0)\, \delta(x-x')\, e(x)\, e^*(x')\, e^{-i\frac{2\pi}{\lambda}\frac{L}{2}\cos\theta_0(x+x')}\, dx\, dx'$$

$$= \int b(x+\theta_0)\, e(x)\, e^*(x)\, e^{-i2\pi\frac{L}{\lambda}\cos\theta_0 x}\, dx$$

Note that the antenna pattern simply weights our measurement, so that the true measurement is convolved with the antenna pattern. For simplicity let $e(x) = \text{rect}(x)$ with appriate limits, and thus

$$V(\theta_0) = \int b(x+\theta_0)\, e^{-i 2\pi \frac{L}{\lambda}\cos\theta_0 x}\, dx$$

If we let $u_0 = \frac{L}{\lambda}\cos\theta_0$, we can rewrite this as

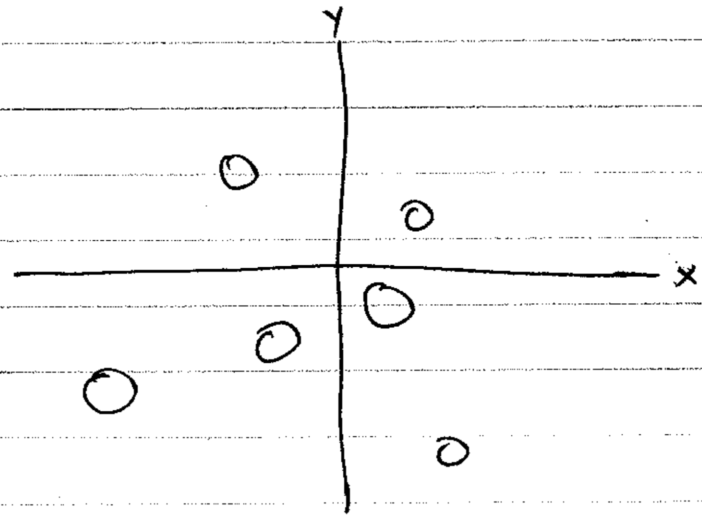
$$V(\theta_0) = \int b(x+\theta_0) e^{-i 2\pi u_0 x}\, dx$$

Therefore the visibility at some direction θ_0 is the Fourier component of the illuminated part of the sky at frequency u_0, which is $\frac{L}{\lambda}\cos\theta_0$. For near vertical viewing, this reduces to $\frac{L}{\lambda}$.

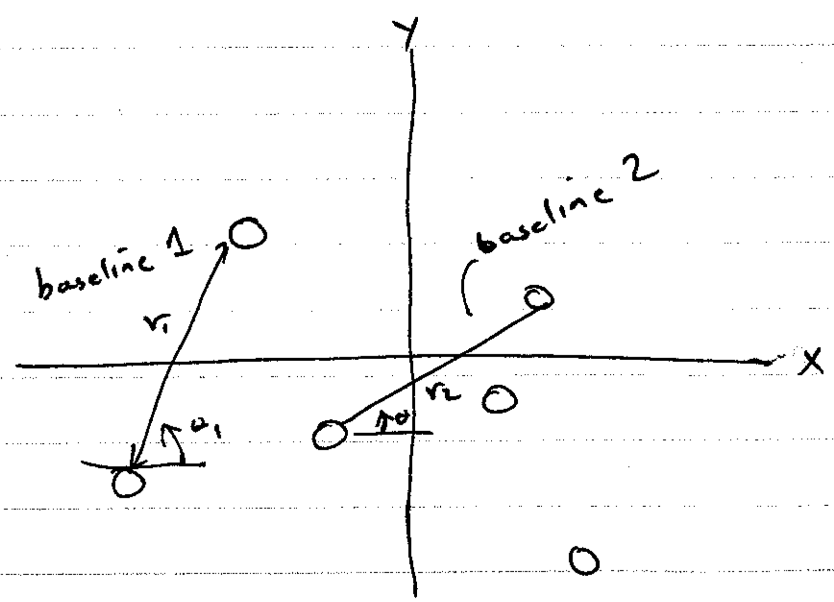
So, the correlating interferometer measures directly the Fourier component corresponding to a spacing $\frac{L}{\lambda}$, without the presence of the near-zero frequency term. We can then populate a synthetic array by either using multiple antennas or multiple measurements at variable spacing.

Extension to 2-D

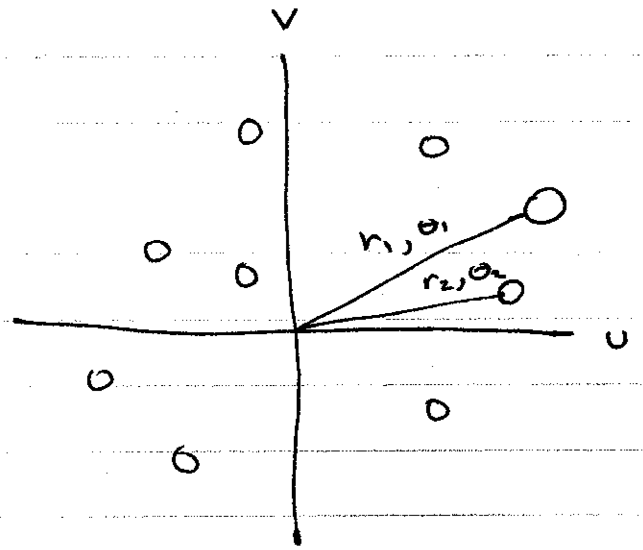
The above examples were all in one dimension, but they generalize to 2-D with no change except to include spacings in 2-D instead of 1-D. For example, suppose we construct a 2-D array such as



Here we would either calculate the array autocorrelation for an adding interferometer, or measure the individual signal products for each pair:

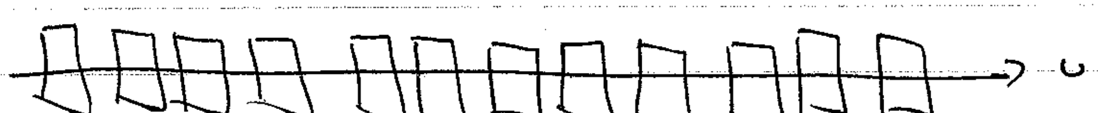


Note that each antenna pair yields one Fourier coeficient, so that we create a sampled version of the u-v plane:

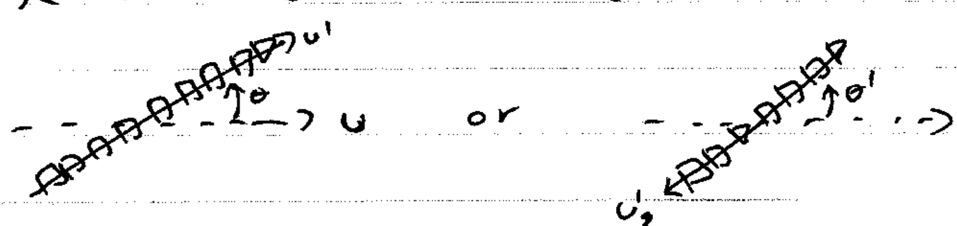


We can then calculate the Fourier transform of these visibilities to reconstruct the image. But note that the u-v plane is sparsely sampled, so that many parts of the spectrum are unmeasured. It is very expensive to fill in the entire array because each antenna is expensive, or at least must be made transportable, which is hard for a large, precise instrument.

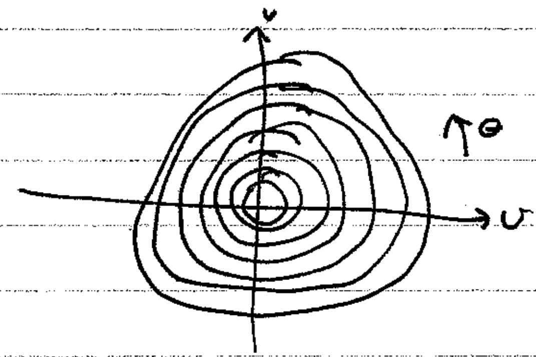
Fortunately, there is an easier and cheaper way to fill in the aperture. Suppose we build a single linear array with variable spacing that has a fairly uniform autocorrelation. Say the autocorrelation looks like this:



Although the spacing may be a bit irregular, most of the u-axis is filled. Now, suppose we built the array on a large turntable that can be rotated:



This array would eventually fill a circle on the u-v plane:



If we make a measurement at many directions θ we can approximate a filled circular aperture. Our achievable resolution is approximately $\frac{\lambda}{u/2}$, if u is the length of the array.

For 1 arc second, we have (at $\lambda = 10$ cm)

$$1'' = \frac{10\text{ cm}}{u/2}$$

$$\frac{1}{3600} \times \frac{\pi}{180} = \frac{10\text{ cm}}{u/2}$$

$$4.85 \times 10^{-6} = 20\text{cm}/u \quad \Rightarrow u = 41\text{ km}$$

Rotating a 41 km array isn't easy, or is it?

We are on a large, rotating turntable, the Earth. Hence if we build and orient a long, multielement array properly the Earth's rotation will generate our circular array in the u-v plane. This was the approach of Ryle (1962), who used the following diagram to illustrate the principle. Observation over twelve hours would suffice to fill the circle completely. (Why 12 and not 24?)

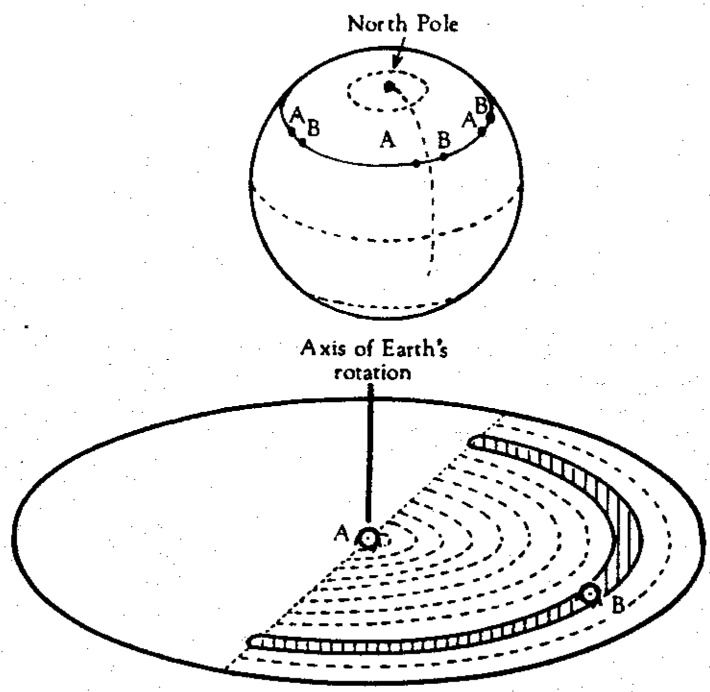


Figure 1.13 Use of earth rotation in synthesis mapping as explained by Ryle (1962). The antennas A and B are spaced on an east–west line. By varying the distance between the antennas from one day to another, and observing for 12 hr with each configuration, it is possible to encompass all the spacings from the origin to the elliptical outer boundary of the lower diagram. Only 12 hr of observing are required, since during the other 12 hr the spacings covered are identical but the positions of the antennas are effectively interchanged. Reprinted by permission

In Ryle's day there was no single line array sufficient to populate the one-interferometer densely enough for fine mapping. So, multiple 12 hr observations were used for two antennas, each at a different spacing. Each spacing filled in one annulus, and the results were combined by computer.

This required a full 2-D synthesis approach to the Fourier transform.