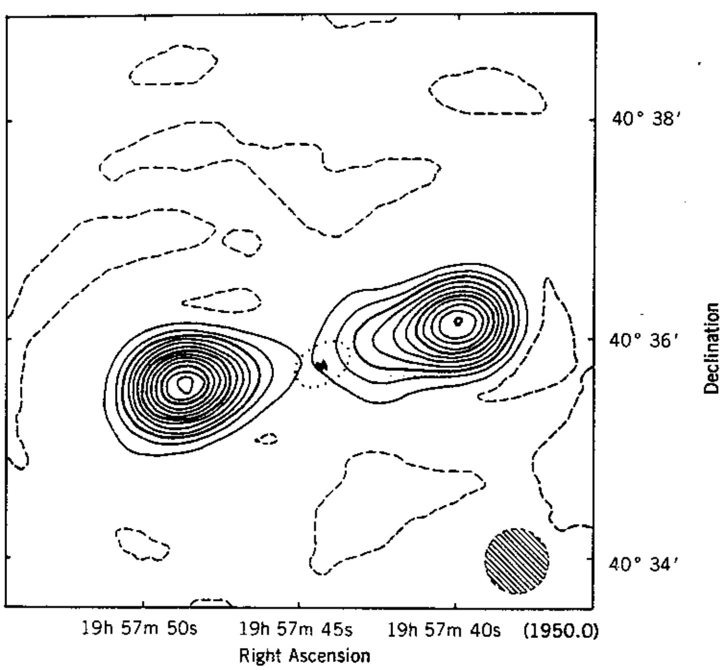
Radio Astronomy

We'll now begin a series of talks involving applications of two-D imaging principles. Some of these require intimate knowledge of the mathematical tools we have developed in class, while others are more practical implementations of basic imaging science.

The first example we will use is that of radio astronomy, which drove much of the development of imaging techniques, and still uses them. After proof in the astronomical context, many of the methods later found their way into medicine and other related fields.

The radio sky

The optical sky can be described as a 2-D array of objects of varying intensities, ~~most~~ many being point objects (the stars) but many having a distribution in space (galaxies and other nebulae). Similarly the radio sky can be thought of as a distribution at radio wavelengths of many objects, some point sources and some distributions.



An early map of the source Cygnus A (Ryle et al., 1965). The circle in the lower right is the beam size, the dotted line is the optical image.

Many, if not most, astronomical objects emit very broadband radiation:

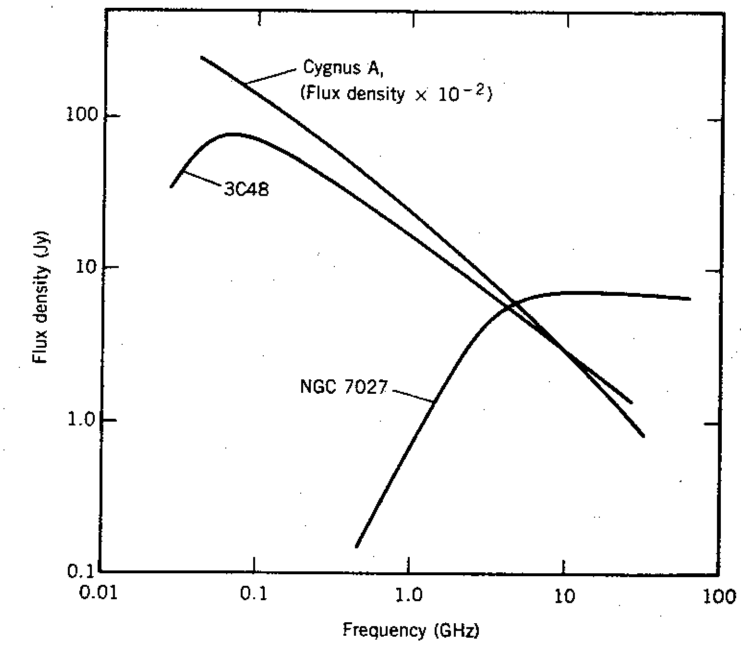


Figure 1.1 Continuum spectra of three discrete sources: Cygnus A, a radio galaxy; 3C48, a quasar; and NGC 7027, an ionized nebula within our galaxy. Data are from Conway, Kellermann, and Long (1963), Kellermann and Pauliny-Toth (1969), and Thompson (1974). One jansky (Jy) = 10^{-26} W m^{-2} Hz^{-1}.

But often narrowband techniques isolating particular spectral lines are called for:

TABLE 1.1 Important Radio Lines

Chemical Name	Chemical Formula	Transition	Frequency (GHz)
Hydrogen	H	$1^2S_{1/2}, F = 1\text{–}0$	1.420
Hydroxyl radical	OH	$^2\Pi_{3/2}, J = 3/2, F = 1\text{–}2$	1.612
	OH	$^2\Pi_{3/2}, J = 3/2, F = 1\text{–}1$	1.665
	OH	$^2\Pi_{3/2}, J = 3/2, F = 2\text{–}2$	1.667
	OH	$^2\Pi_{3/2}, J = 3/2, F = 2\text{–}1$	1.721
Formaldehyde	H_2CO	$1_{10}\text{–}1_{11}$, six F-transitions	4.830
Water	H_2O	$6_{16}\text{–}5_{23}$, five F-transitions	22.235
Ammonia	NH_3	1, 1–1, 1, eighteen F-transitions	23.694
	NH_3	2, 2–2, 2, seven F-transitions	23.723
	NH_3	3, 3–3, 3, seven F-transitions	23.870
Silicon monoxide	SiO	$v = 2, J = 1\text{–}0$	42.820
	SiO	$v = 1, J = 1\text{–}0$	43.122
	SiO	$v = 1, J = 2\text{–}1$	86.243
Hydrogen cyanide	HCN	$J = 1\text{–}0$, three F-transitions	88.631
Formyl ion	HCO^+	$J = 1\text{–}0$	89.189
Carbon monoxide	$^{12}C^{18}O$	$J = 1\text{–}0$	109.782
Carbon monoxide	$^{13}C^{16}O$	$J = 1\text{–}0$	110.201
Carbon monoxide	$^{12}C^{16}O$	$J = 1\text{–}0$	115.271
Carbon monoxide	$^{12}C^{16}O$	$J = 2\text{–}1$	230.538
Carbon	C	$^3P_1\text{–}^3P_0$	492.162

The quantity we measure is generally called flux density, and has units of watts per square meter per Hertz ($W\,m^{-2}Hz^{-1}$). It is, however, an intensity quantity so that the power pattern of the antenna is of interest.

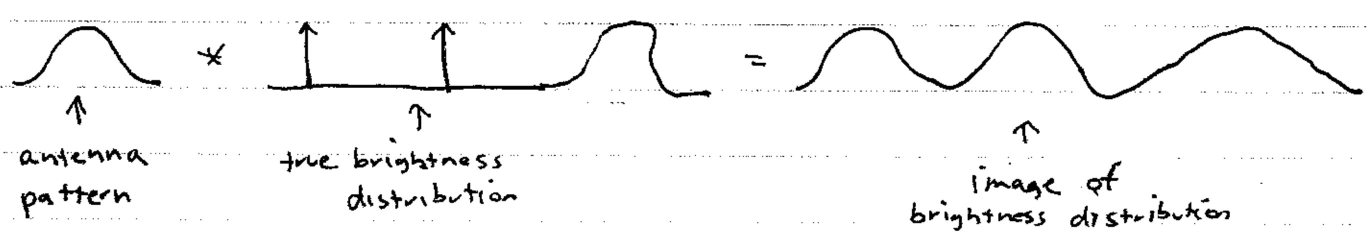
The celestial coordinate system is quite often given in terms of angles right ascension α and declination δ, which are analogous to longitude and latitude on Earth. The zero latitude is the celestial equator, the zero longitude is taken as where the sun's orbit intersects the celestial equator as the sun transits from below to above it. This point is also called the vernal equinox. For historical reasons, α is in units of hrs:min:secs while δ is in Deg/min/secs.

Let's suppose we image the sky with an antenna with a power pattern $p(\theta, \phi)$, which we for now assume is small compared to the sky. In other words, we assume the sky is "flat" in a region of interest. Let the true radio distribution have brightness $b(\alpha, \delta)$, then the measured image $i(\alpha, \delta)$ is

$$i(\alpha, \delta) = p(\alpha, \delta) ** b(\alpha, \delta)$$

The image is the true brightness convolved with the antenna power pattern.

A 1-D example:



The image is thus the convolution of the true brightness with the power pattern of the antenna.

What do we know about power patterns and apertures? Let's derive some aperture-pattern relations.

Field Apertures and Power Patterns

We have already examined the Fraunhofer diffraction formula relating antenna far fields to the excitation aperture. We had

$$g(\theta) = \int_{-\infty}^{\infty} f(x)\, e^{-i 2\pi x \frac{\sin\theta}{\lambda}}\, dx$$

Within scaling, this is the Fourier transform relation

$$f(x) \supset g(\theta)$$

where $g(\theta)$ is the amplitude of the fields in the far field. This amplitude represents the electric field quantity.

We found that the radio image was related to the power pattern, not the amplitude pattern. Denoting the power pattern by $p(\theta)$

$$p(\theta) = g(\theta)\, g^*(\theta) = |g(\theta)|^2$$

We can still relate the power pattern to the original aperture if we use the autocorrelation theorem:

$$f(x) \star f(x) \supset g(\theta)\, g^*(\theta) = p(\theta)$$

Substituting back into the image convolution in 1-D form:

$$i(\theta) = p(\theta) * b(\theta)$$

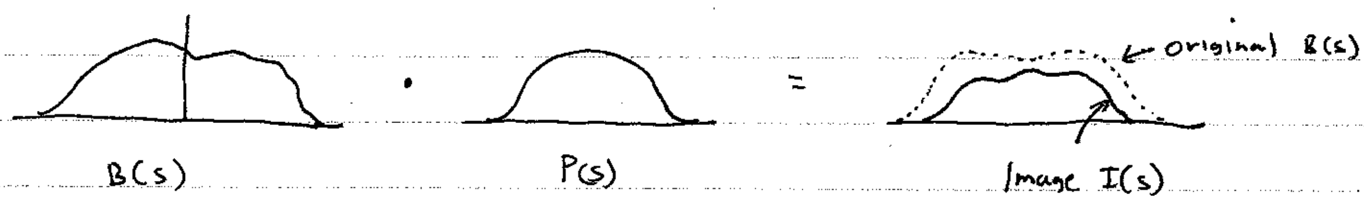
In transform space

$$I(s) = P(s) \cdot B(s)$$

$$I(s) = f(x) \star f(x) \cdot B(s)$$

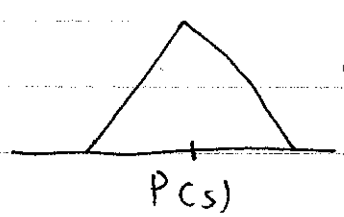
The transfer function, $P(s)$, is thus seen to be the autocorrelation of the aperture illumination function. How can we understand this physically?

First, look at the previous equation pictorially:

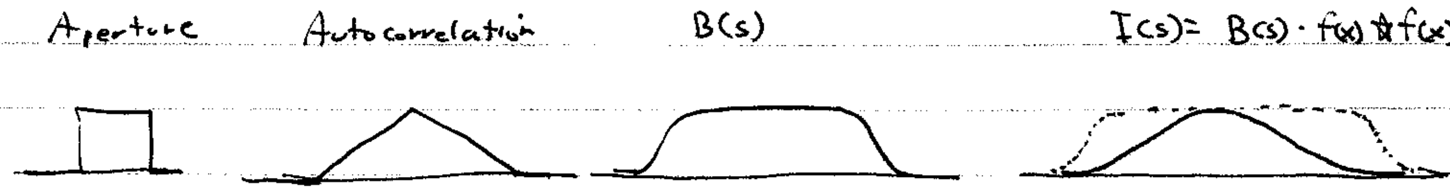


Hence, $B(s)$ is "weighted" by $P(s)$. Suppose $P(s)$ were a rect ( ) function. Then $B(s) \cdot P(s) = I(s)$ would be a low-pass filtered version of the true distribution, where $P(s)$ defines the filter shape and cutoff.

In a more realistic case, let the illumination of the aperture be rect (x), a uniformly illuminated rectangular aperture. Its autocorrelation would be a triangle function $\Lambda(x)$, so $P(s)$ would look like



where the width of the triangle is twice the width of the rect x aperture. So, we might have the following situation

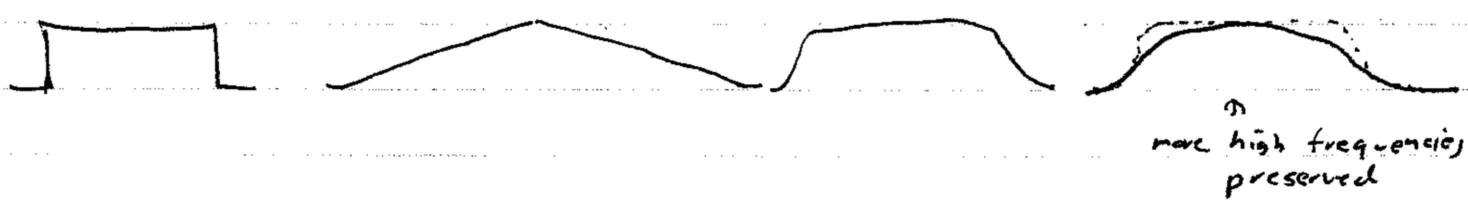


In usual antenna theory, we might have the following:

Antenna aperture: L meters
Wavelength: λ
Antenna beamwidth: $\frac{\lambda}{L}$ radians

Hence the sky is "smeared" to an angular resolution of $\frac{\lambda}{L}$ by the antenna.

What if we double the aperture size? Then the beamwidth becomes $\frac{\lambda}{2L}$, and "smearing" is reduced by a factor of 2. In transform notation



In this trivial case it was just as easy to work out the result with the equations as with transform representation. But what if we consider more complicated apertures?

Interferometers

Clearly we can achieve higher resolution, and hence more accurate sky mapping, by increasing aperture size. However, at some point this becomes impractical. Consider the Big Dish here at Stanford. It is 150 feet, or 46 meters, across. Let's say that it is constructed accurately enough to be an efficient antenna at wavelengths of 10 cm or more. What is its beamwidth?

The aperture would be described by

$$\mathrm{rect}\left(\frac{x}{460}\right)$$

where the units are wavelengths. (We often use normalized units this way in antenna calculations to make them simpler.) What is the antenna pattern? The field pattern is the transform of the aperture

$$\mathrm{rect}\left(\frac{x}{460}\right) \underset{H}{\supset} 460^2 \,\mathrm{jinc}(460\,\theta)$$

Of course all of this is multiplied by gain and other constants. The power pattern is proportional to

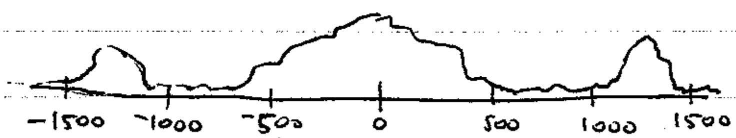
$$p(\theta) \propto \mathrm{jinc}^2(460\,\theta)$$

and since the first null of the jinc is at 1.22, the peak to null width of the pattern is at $\theta = \frac{1.22}{460}$ radians $= 2.65\times10^{-3}$ rad or 0.15°. While this beam resolves the sun or the moon, each about $\frac{1}{2}^\circ$, it could not resolve the 7.2×10^{-4} radian distance between the two sources in Cygnus A.

What is the equivalent limit in transform analysis? We begin with the same aperture $\mathrm{rect}\left(\frac{x}{460}\right)$.

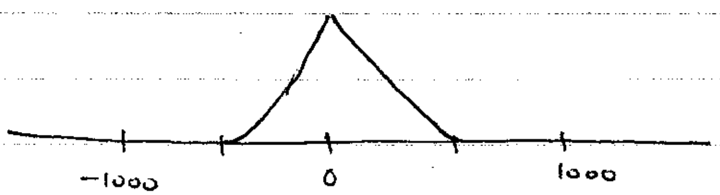
$$\text{rect}\ \frac{x}{460}\ \star\star\ \text{rect}\ \frac{x}{460} = \text{chat}\ \frac{s}{460}$$

Let's say our true brightness has a significant amount of information at $\frac{1}{7.2\times10^{-4}}$ cycles/radian = 1375 cycle/rad



B(s)

↑
Information on
spacing of Cygnus A

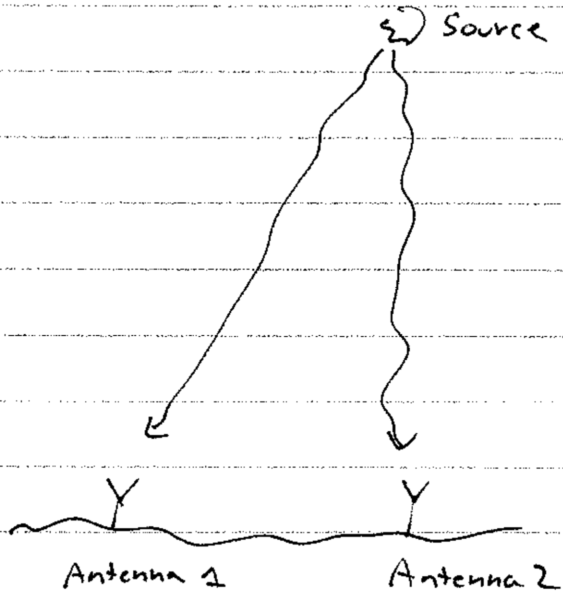


$\text{chat}\left(\frac{s}{460}\right)$

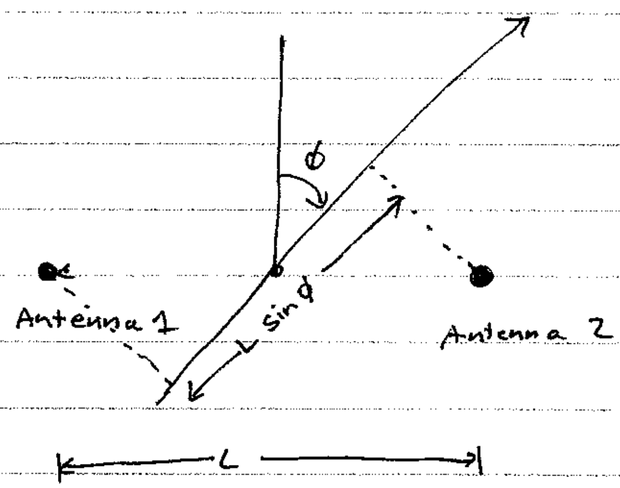
Note that the chat has no response at $s = 1375$, so the spacing component of the source Cygnus A will not be observed by the Big Dish. However, the lower frequency part, centered around 0, will indeed be visible.

How can we map such high-frequency distributions? We could build a bigger dish (expensive), or a more accurate dish that is ~~flatt~~ smoother so we could use a shorter wavelength (also expensive). Fortunately there is another alternative.

Suppose we construct a second antenna at some distance away from the first antenna, and sum their signals. The situation:



Geometrically, the signal received at angle ϕ from the midpoint direction can be expressed using the following:



The signal to antenna 1 travels $L \sin\phi$ further than the signal to antenna 2. If $E_o(\theta)$ describes the field pattern received by a similar antenna located at the midpoint, the total signal can be expressed as

$$E = E_0(\theta)\, e^{i \frac{2\pi}{\lambda} \frac{L}{2} \sin\phi} + E_0(\theta)\, e^{-i \frac{2\pi}{\lambda} \frac{L}{2} \sin\phi}$$

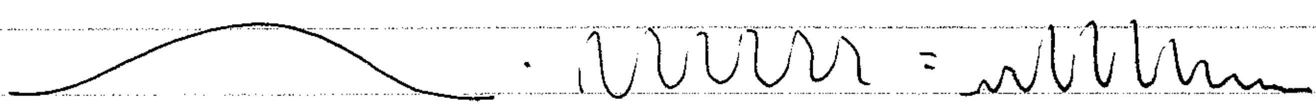
$$= E_0(\theta) \cos\left[\frac{2\pi}{\lambda} \frac{L}{2} \sin\phi\right]$$

If ϕ is small, as it is in the cases we are concerned with,

$$E = E_0(\theta) \cos\left(\frac{2\pi}{\lambda} \frac{L}{2} \phi\right)$$

The total effective antenna response thus has two components, a single element pattern $E_0(\theta)$ and an array pattern $\cos \frac{2\pi}{\lambda} \frac{L}{2} \phi$. This kind of analysis is usually referred to as array theory for antennas.

What does this pattern look like? For example:



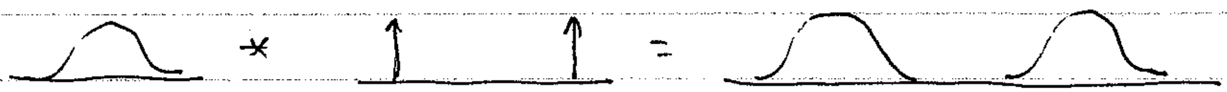
$E_0(\theta)$ $\cos \frac{2\pi}{\lambda} \frac{L}{2} \phi$

Broad, ~ Gaussian say

Interferometer pattern sensitive to a particular spatial frequency

We can also understand this from our transform approach. We can represent the array as a convolution of a single element with an array. Letting the single element have a Gaussian beam as above:

the array would be



Aperture distribution for 1 antenna — Two antenna locations — Effective aperture.

Since the pattern is the transform of the aperture, we simply calculate the transforms (patterns) of the 1-antenna aperture and the array and multiply, obtaining the same result as before:

$$E_{interferometer} = E_0(\theta) \cdot \cos(\quad)$$

How does this relate to our transfer function theory for what we can measure? First, get the autocorrelation of the aperture:

This is the passband of the "adding" interferometer. It is sensitive to the low freqency parts of a distribution plus some high-frequency components.

Next: how to measure the complete distribution.