More on Hankel transforms

Rather than derive a bunch of transforms and theorems, we'll just list the following table:

Table 9-1 Table of Hankel transforms.

$f(r)$	$F(q) = 2\pi \int_0^\infty f(r) J_0(2\pi q r) r dr$
$f(ar)$	$a^{-2} f(q/a)$
$f ** g$	FG
$r^2 f(r)$	$-\nabla^2 F$
$\mathrm{rect}\, r$	$\mathrm{jinc}\, q$
$\delta(r-a)$	$2\pi a J_0(2\pi a q)$
$e^{-\pi r^2}$	$e^{-\pi q^2}$
$r^2 e^{-\pi r^2}$	$\pi^{-1}(\pi^{-1} - q^2) e^{-\pi q^2}$
$(1+r^2)^{-1/2}$	$q^{-1} e^{-2\pi q}$
$(1+r^2)^{-3/2}$	$2\pi e^{-2\pi q}$
$(1-4r^2)\, \mathrm{rect}\, r$	$J_2(\pi q)/\pi q^2$
$(1-4r^2)^\nu\, \mathrm{rect}\, r$	$2^{\nu-1} \nu! J_{\nu+1}(\pi q)/\pi^\nu q^{\nu+1}$
r^{-1}	q^{-1}
e^{-r}	$2\pi(4\pi^2 q^2 + 1)^{-3/2}$
$r^{-1} e^{-r}$	$2\pi(4\pi^2 q^2 + 1)^{-1/2}$
$^2\delta(x, y)$	1

Most Hankel theorems follow from 2-D Fourier theorems with circular symmetry conditions. For example, scaling in x must be accompanied by equal scaling in y, and so forth.

The convolution theorem has meaning only when we remember that both $f(r)$ and $F(q)$ are taken to be 2-D functions defined on the x-y plane.

Several common transform pairs are given above. Several of these we have previously introduced, such as rect - jinc, $\delta - 1$, $\delta(r-a) - 2\pi a J_0(2\pi a q)$. Others are obvious interpretations of our 2-D pairs, like $e^{-\pi r^2} - e^{-\pi q^2}$.

Computing Hankel transforms

Because the Hankel transform is a 1-D transform, its computation requires less time than the full 2-D Fourier transform. Thus we can use a simple numerical integration to evaluate it, as for example this "program" given by Bracewell:

```
HANKEL TRANSFORM
DEF FNf(r)=EXP(-PI*(r/7)^2        Function definition
dr=0.1                            Step in r
FOR q=0 TO 0.25 STEP 0.05
   k=2*PI*q
   s=0
   FOR r=dr/2 TO 14 STEP dr
      s=s+FNf(r)*FNJ0(k*r)* r
   NEXT r
   PRINT q;2*PI*s*dr
NEXT q
END
```

For long arrays or when high accuracy is needed, this routine may be too slow because of the many calls to the $J_0(\ )$ function. It is often faster to implement the Hankel transform by first computing an Abel transform and follow with an FFT, as we will note below.

Everything you wanted to know about the jinc function

Because the jinc function is so fundamental to 2-D imaging, we'll consider several of its properties in detail.

In 1-D, we know the sinc function consists of all frequencies up to some cutoff equally weighted and added together. Similarly in 2-D the jinc function consists of all 2-D corrugations up to some cutoff and summed.

Begin with the definition:

$$\text{jinc}\, x = \frac{J_1(\pi x)}{2x}$$

The series expansion for jinc is derived from that of the $J_1(\ )$:

$$\text{jinc } x = \frac{\pi}{4} - \frac{\pi^3}{2^5}x^2 + \frac{\pi^5}{2^8\cdot 3}x^4 - \frac{\pi^7}{2^{12}\cdot 3}x^6 + \frac{\pi^9}{2^{16}\cdot 3^2\cdot 5}x^8 - \cdots$$

Should be 3^2, I think, but this is how it appears in text

For large arguments, this is an asymptotic expression similar to that for $J_0(\ )$:

$$\text{jinc}(x) \approx \frac{1}{\sqrt{2x^3\pi^2}} \cos\left[\pi\left(x-\tfrac{3}{4}\right)\right] \qquad x > 3$$

Note that this fails for $x<0$, but for jinc we are interested in positive radius only. For "negative" radius, we use the positive value by symmetry.

The asymptotic behavior of jinc is that it falls off relatively rapidly with r, indicating that its transform possesses a single finite discontinuity ~~shar~~. The $\text{jinc}^2(\ )$ function falls off more quickly still, as its transform chat(q) is smoother yet, possessing no discontinuities.

Zeroes of jinc. The zeroes of the jinc function are approximately

nth zero	1	2	3	4	$\cdots$	n
location	1.220	2.233	3.238	4.241		$\sim n+\frac{1}{4}$

Derivative.

$$\frac{\partial}{\partial x}\text{jinc}(x) = \frac{\pi}{2\pi}J_0(\pi x) - \frac{1}{x^2}J_1(\pi x) = -\frac{\pi}{2x}J_2(\pi x)$$

which again follows from Bessel function derivative rules.

Sidelobes - Maxima and minima

Location	1.635	2.679	3.699	4.710
Value	-0.104	0.051	-0.031	0.022

Integral: The area under the jinc is unity:

$$\int_{-\infty}^{\infty} \operatorname{jinc}(x)\,dx = 1$$

How might we see this immediately?

Half peak and half-power point:

$$\operatorname{jinc}(0.70576) = \frac{1}{2}\operatorname{jinc} 0 = \frac{\pi}{8}$$

$$\theta_{\text{half-power}} = 0.51456$$

$$\operatorname{jinc}\theta_{\text{half-power}} = \frac{1}{\sqrt{2}}\operatorname{jinc} 0 = 0.55536$$

Fourier transform:

$$\int_{-\infty}^{\infty} \operatorname{jinc} x\, e^{-i2\pi sx}\,dx = \sqrt{1-4s^2}\ \operatorname{rect} s$$

Note this is the 1-D transform of jinc, not 2-D.

Hankel transform: We know this is a rect:

$$\int_{0}^{\infty} \operatorname{jinc} r\ J_0(2\pi q r)\cdot 2\pi r\,dr = \operatorname{rect} q$$

Abel transform: We will define the Abel transform below, but for now consider it a line integral through the jinc:

$$\text{Abel transform} = 2\int_x^\infty \frac{\text{jinc } r}{\sqrt{r^2-x^2}}\, r\,dr = \text{sinc } x$$

Volume under jinc:

$$\iint_{-\infty}^{\infty} \text{jinc}\left(\sqrt{x^2+y^2}\right) dx\,dy = \int_0^\infty \text{jinc } r \; 2\pi r\,dr = 1$$

Again, how could we get this immediately?

Autocorrelation: The jinc is its own autocorrelation.

$$\iint_{-\infty}^{\infty} \text{jinc}\left((\xi^2+\eta^2)^{1/2}\right) \text{jinc}\left(((\xi+x)^2+(\eta+y)^2)^{1/2}\right) d\xi\,d\eta = \text{jinc}\sqrt{x^2+y^2}$$

Hankel transform of $\text{jinc}^2 r$ is the chat q.

$$\int_0^\infty \text{jinc}^2 r \; J_0(2\pi q r)\, 2\pi r\,dr = \text{chat } q = \tfrac{1}{2}\left(\cos^{-1}|q| - |q|\sqrt{1-q^2}\right) \text{rect}\left(\tfrac{q}{2}\right)$$

Autocorrelation of unit pillbox is chat

2-D Fourier transform of jinc^2 is chat

More relations are given in the text on pages 361-363.

Relation between Fourier, Hankel, and Abel transforms

Beginning with a function $f(r)$, we can relate various transforms according to the following graphical depiction:

$$
\begin{tikzcd}
f(r) \arrow[rr, leftrightarrow, "\text{Fourier Transform}"] \arrow[d, "\text{Abel Transform}"'] \arrow[drr, leftrightarrow, "\text{Hankel Transform}" description] & & F(s) \\
A(x) \arrow[rr, leftrightarrow, "\text{Fourier Transform}"'] & & H(q) \arrow[u, "\text{Abel Transform}"']
\end{tikzcd}
$$

The arrows show that the Hankel and Fourier transforms are invertible, whereas the Abel transform is not. The various arguments show how we interpret the transforms. The Fourier relations shown here are all 1-D transforms. How all of these are used to help evaluate the various transforms awaits a discussion of the Abel transform.

The Abel transform

Abel was a mathematician whose name has been associated with a transform giving the line integral through a circularly symmetric function. Because of the symmetry, the line integral is the same in all directions and depends only on distance from the origin. The variable is given as x and we can think of it as the abscissa in the (x,y) plane over which

The radial coordinate r is defined.

In this sense the Abel transform $f_A(x)$ is the projection in the y-direction of the symmetric function.

An example is the circular pillbox rect(r):

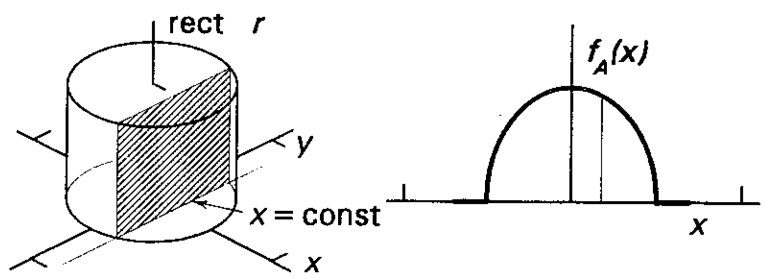
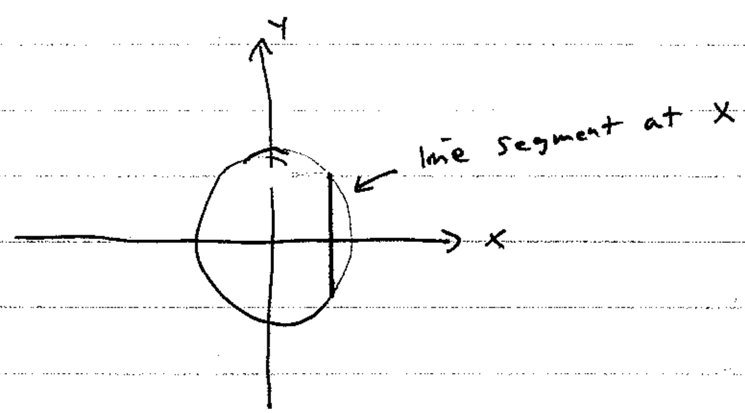


Figure 9-13 The area of the shaded cross section is the Abel transform of the function of r for the particular value of x chosen. In the case of rect r the Abel transform $f_A(x)$ is the semi-ellipse $\sqrt{1-4x^2}$, $|x| < \frac{1}{2}$.

From the top looking down, we have



and the projection is simply the length of the line segment at X. Note that for $|x| > \frac{1}{2}$ the transform, or projection, is zero. If the function $f(r)$ has varying strength with r, it would be incorporated into the integral and a weighted segment would be used.

Thus, we can define the Abel transform by

$$f_A(x) = \int_{-\infty}^{\infty} f\left(\sqrt{x^2+y^2}\right) dy$$

So, what is the Abel transform of rect r?

$$f_A(x) = \int_{-\infty}^{\infty} \text{rect}\, \sqrt{x^2+y^2}\, dy$$

$$= \int_{-(\frac{1}{4}-x^2)^{1/2}}^{(\frac{1}{4}-x^2)^{1/2}} dy \ \text{rect}\, x$$

$$= 2\sqrt{\tfrac{1}{4}-x^2}\ \text{rect}\, x$$

$$= \sqrt{1-4x^2}\ \text{rect}\, x$$

Usually, however, we formulate the Abel transform in terms of the circular coordinate system because the functions are circularly symmetric. It also usually means the limits on the integrals are easier to determine.

In this coordinate system

$$f_A(x) = 2\int_x^{\infty} \frac{f(r)}{\sqrt{r^2-x^2}}\, r\, dr$$

which follows from $\frac{dr}{dy} = \sin\theta = \frac{\sqrt{r^2-x^2}}{r}$. Note that the minimum value of r is the value of x at which we evaluate the line integral, and $\int_{-\infty}^{\infty} dy \to 2\int_x^{\infty} \frac{dy}{dr}\cdot dr$, where the 2 comes from using contributions above and below the x-axis.

Abel transform theorems

Similarity: $f(ar) \underset{A}{\supset} a^{-1} f_A(ax)$

Note that the arguments are ar and ax; no a inversion.

~~Superposition~~

Superposition:

$$f(r) + g(r) \underset{A}{\supset} f_A(x) + g_A(x)$$

Convolution theorem:

$$f(r) ** g(r) \underset{A}{\supset} \left[f_A(r) ** g_A(r) \right]_{\theta = \text{constant}}$$

Note that the convolution of $f_A(r)$ and $g_A(r)$ is a 2-D convolution, which is then evaluated along a line of constant θ.

Area integral conservation:

$$2\pi \int_0^\infty f(r)\, r\, dr = \int_{-\infty}^{\infty} f_A(x)\, dx$$

This follows from $f_A(x)$ being the projection of $f(r)$.

Central value theorem:

$$f_A(0) = 2 \int_0^\infty f(r)\, dr$$

Inversion. While the transform itself is not reversible, there is an inversion formula relating the derivative $f_A'(x)$ to $f(r)$

$$f(r) = -\frac{1}{\pi} \int_r^\infty \frac{f_A'(x)}{\sqrt{x^2 - r^2}}\, dx$$

We sometimes need this formula for numerical inversion when line-integrated data of a function are available and the function itself is inaccessible.

Some Abel transforms

Table 9-2 Table of Abel transforms. For compactness rect x is written $\Pi(x)$.

$f(r)$		$f_A(x) = 2\int_x^\infty (r^2 - x^2)^{-1/2} f(r) r\,dr$	
$\Pi(r/2a)$	Disk	$2(a^2 - x^2)^{1/2}\Pi(x/2a)$	Semiellipse
$(a^2 - r^2)^{-1/2}\Pi(r/2a)$		$\pi\Pi(x/2a)$	Rectangle
$(a^2 - r^2)^{1/2}\Pi(r/2a)$	Hemisphere	$\frac{1}{2}\pi(a^2 - x^2)\Pi(x/2a)$	Parabola
$(a^2 - r^2)\Pi(r/2a)$	Paraboloid	$\frac{4}{3}(a^2 - x^2)^{3/2}\Pi(x/2a)$	
$(a^2 - r^2)^{3/2}\Pi(r/2a)$		$\frac{3\pi}{8}(a^2 - x^2)^2\Pi(x/2a)$	
$(1 - \lvert r\rvert)\Pi(r/2)$	Cone	$[(a^2 - x^2)^{1/2} - (x^2/a)\cosh^{-1}(a/x)]\Pi(x/2a)$	
$\cosh^{-1}(a/r)\Pi(r/2a)$		$\pi a(1 - \lvert r/a\rvert)\Pi(r/2a)$	Triangle
$\delta(r - a)$	Ring impulse	$2a(a^2 - x^2)^{-1/2}\Pi(x/2a)$	
$\exp(-\pi r^2)$	Gaussian	$W\exp(-\pi x^2/W^2)$	Gaussian
$\exp(-r^2/2\sigma^2)$	Normal	$\sqrt{2\pi}\sigma\exp(-x^2/2\sigma^2)$	Normal
$r^2\exp(-r^2/2\sigma^2)$		$\sqrt{2\pi}\sigma(x^2 + \sigma^2)\exp(-x^2/2\sigma^2)$	
$(r^2 - \sigma^2)\exp(-r^2/2\sigma^2)$		$\sqrt{2\pi}\sigma x^2\exp(-x^2/2\sigma^2)$	
r^{-2}		π/x	
$(a^2 + r^2)^{-1}$		$\pi(a^2 + x^2)^{-1/2}$	
$J_0(2\pi ar)$	Bessel	$(\pi a)^{-1}\cos 2\pi ax$	Cosine
$2\pi[r^{-3}\int_0^r J_0(r)dr - r^{-2}J_0(r)]$		$\operatorname{sinc}^2 x$	
$\delta(r)/\pi\lvert r\rvert$		$\delta(x)$	Impulse
$2a\operatorname{sinc}(2ar)$		$J_0(2\pi ax)$	Bessel
$\frac{1}{2}r^{-1}J_1(2\pi ar)$		$\operatorname{sinc} 2ax$	
$\operatorname{jinc} r$		$\operatorname{sinc} x$	

Hankel transform without Bessel functions

Now we can return to the idea of evaluating Hankel transforms without resorting to the calculation of Bessel functions. We had

$$\begin{tikzcd}
f(r) \arrow[r, leftrightarrow, "\text{Fourier}"] \arrow[d, "\text{Abel}"'] \arrow[dr, leftrightarrow, "\text{Hankel}"] & F(s) \\
f_A(x) \arrow[r, leftrightarrow, "\text{Fourier}"] & H(q) \arrow[u, "\text{Abel}"']
\end{tikzcd}$$

We can thus calculate the Hankel transform by first using the Abel transform, followed by a Fourier transform.

Examples:

$$\begin{tikzcd}
\operatorname{jinc} r \arrow[r, leftrightarrow, "F"] \arrow[d, "A"'] & \sqrt{1-4s^2}\, \operatorname{rect} s \\
\operatorname{sinc} x \arrow[r, leftrightarrow, "F"] & \operatorname{rect} q \arrow[u, "A"'] \arrow[ul, leftrightarrow, "H"]
\end{tikzcd}$$

$$\begin{tikzcd}
\operatorname{rect} r \arrow[r, leftrightarrow, "F"] \arrow[d, "A"'] & \operatorname{sinc} s \\
\sqrt{1-4x^2}\, \operatorname{rect} x \arrow[r, leftrightarrow, "F"'] & \operatorname{jinc} q \arrow[u, "A"'] \arrow[ul, leftrightarrow, "H"]
\end{tikzcd}$$