Rotational Symmetry

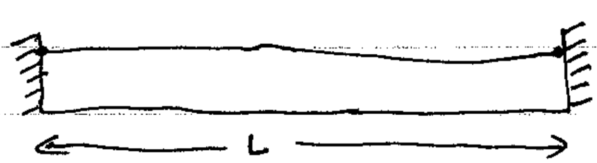
Very often in imaging we are dealing with components exhibiting circular symmetry: lenses are round, as well as many objects we wish to image. In these cases the dependence on θ is not needed - only the dependence on r counts.

For objects described in Cartesian space, it was natural to use the independent variables x and y, which led to expressions of transforms in terms of sines and cosines along the two cardinal directions. In circular symmetry, as we shall see, Bessel functions play the role of the sines and cosines, and form orthonormal basis functions we can use to express the functions in transform space.

We have already introduced the circular pillbox function rect(r) and its transform the jinc function. Let's look at these again in another perspective to try and obtain some intuitive feeling for the nature of these functions.

Bessel functions.

As stated above, Bessel functions are to circularly symmetric 2-D geometries what sines and cosines are to 1-D geometries. Consider a taut string of length L:



Pluck the string and it will vibrate. Solution of the differential equation shows that any periodic motion of the string can be expressed as a sum of sinusoids

$$f(x) = \sum a_i \cos 2\pi \frac{x \cdot i}{L} + b_i \sin 2\pi \frac{x \cdot i}{L}$$

In fact we may define sine and cosine as the functions satisfying the above. Similarly, if we stretch a membrane around a circular aperture we can describe the motion by a series of Bessel functions.

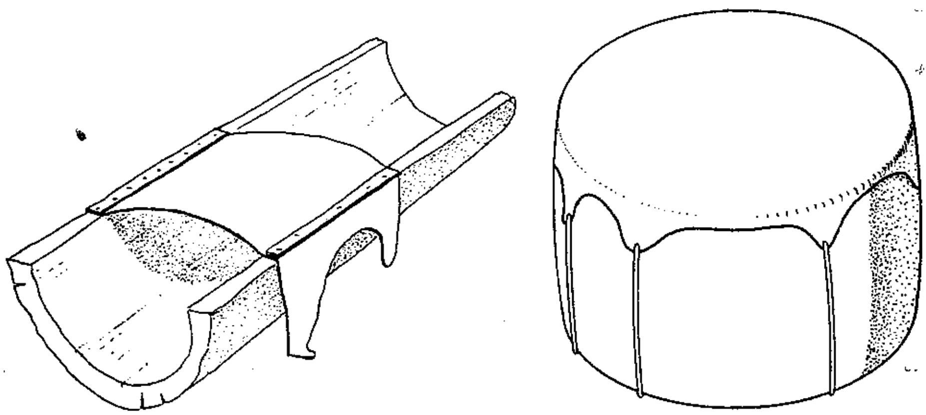


Figure 9-5 The fundamental mode of a laterally stretched membrane is cosinusoidal (left), while the corresponding mode of a circular drum is a zero-order Bessel function.

In fact, the correspondence is closer if we try to think of the drum membrance as motion containing corregations in all ~~dimen~~ directions at once:

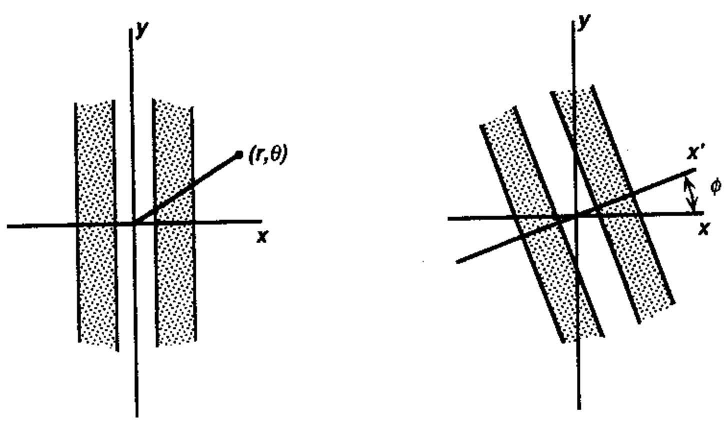


Figure 9-4 The simple corrugation $A \cos 2\pi ux$ (left) becomes $A \cos(2\pi ur \cos\theta)$ in polar coordinates, and the rotated corrugation (right) becomes $A \cos[2\pi ur \cos(\theta - \phi)]$.

If we consider the set of all possible corrugations rotated by Θ, a single corrugation might be expressed as

$$f(x,y) = \cos(2\pi u x)$$

In polar coordinates, since $x = r\cos\theta$

$$f(x,y) = \cos(2\pi u r \cos\theta)$$

If this corrugation is rotated by ϕ, then

$$f(x,y)_{\text{rotated}} = \cos(2\pi u r \cos(\theta - \phi))$$

Now, let's assume that this corrugation occurs in all directions with equal amplitude (circular symmetry). Then we can integrate the above over ϕ; and obtain the sum

$$\int_0^{2\pi} \cos(2\pi u r \cos(\theta - \phi))\, d\phi$$

Now, since we are evaluating over all ϕ's once, a little thought reveals that the integral is indepdent of θ, so let's take it to be 0. Then the superposition of the corrugation at all angles becomes

$$\int_0^{2\pi} \cos(2\pi u r \cos\phi)\, d\phi$$

If we had calculated the angular average rather than the sum we would have found

$$\frac{1}{2\pi}\int_0^{2\pi} \cos(2\pi u r \cos\phi)\, d\phi = J_0(r)$$

or the zero-order Bessel function of the first kind. So Bessel functions are to circular geometry what our sinusoids were to rectangular geometry.

There are other definitions of the Bessel function as well. Two mathematical definitions are the series

$$J_0(r) = 1 - \frac{r^2}{4} + \frac{r^4}{64} - \frac{r^6}{2304} + \cdots$$

and the solution to the differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0$$

This equation also has solution $Y_0(x)$, the Bessel function of the second kind.

If x is large, we have the useful approximation

$$J_0(r) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{1}{4}\pi\right) \qquad (r \text{ large})$$

or a slowly decreasing cosine ~~for~~ shifted by 90°. This is helpful when we need fast calculations for large arguments, as series expansions can be slow and inaccurate for large arguments. For small arguments, the following works well:

for $-3 \le x \le 3$:

$$J_0(x) = 1 - 2.24999\,97(x/3)^2 + 1.26562\,08(x/3)^4 - .31638\,66(x/3)^6 + .04444\,79(x/3)^8 - .00394\,44(x/3)^{10} + .00021\,00(x/3)^{12} + \epsilon, \quad |\epsilon| < 5 \times 10^{-8}.$$

Water in a circular tank sloshing, or waves propagating in a circular waveguide have Bessel function expansions. Some elementary Bessel functions, as well as the sinc and jinc functions, are shown below for reference:

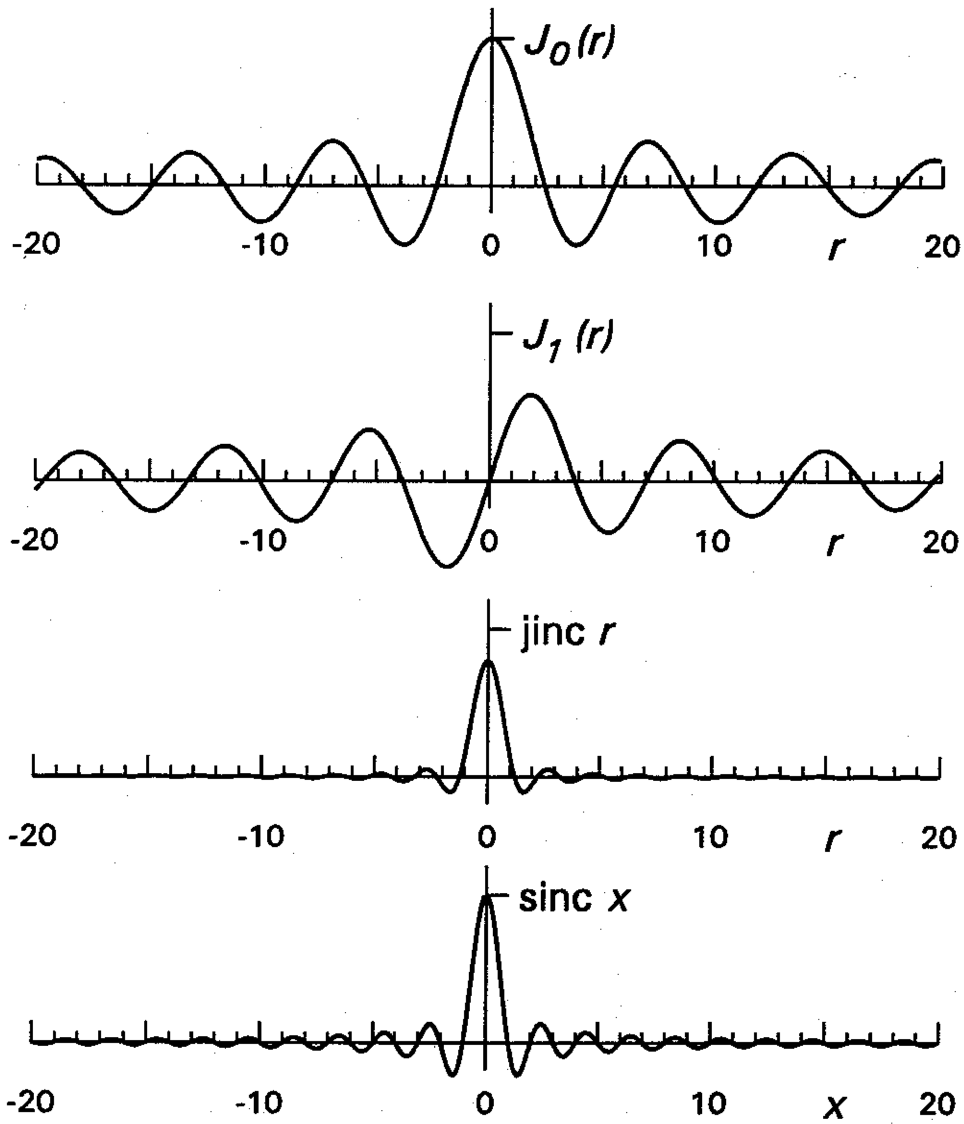


Figure 9-1 The functions $J_0(r)$, $J_1(r)$, jinc r, and sinc x.

The Hankel Transform

The Hankel Transform is the 2-D Fourier transform of a circularly symmetric function expressed as a function of the radius variable only. This is the assumption of circular symmetry.

Consider a function $f(x,y)$ that can be expressed as $f(\sqrt{x^2+y^2})$ or $f(r)$. Its transform, using the 2D Fourier approach, would be:

$$F(u,v) = \iint_{-\infty}^{\infty} f(r)\, e^{-i2\pi(ux+vy)}\, dx\, dy$$

This can of course be evaluated in Cartesian space, but the evaluation is complicated for even simple functions. For example, let $f(r) = K \operatorname{rect}\left(\frac{r}{2a}\right)$, a pillbox of radius a and height K. Then

$$F(u,v) = K \int_{-a}^{a} dy \int_{-(a^2-y^2)^{1/2}}^{(a^2-y^2)^{1/2}} e^{-i2\pi(ux+vy)}\, dx$$

$$= K \int_{-a}^{a} e^{-i2\pi yv}\, dy \int_{-(a^2-y^2)^{1/2}}^{(a^2-y^2)^{1/2}} e^{-i2\pi ux}\, dx$$

$$= K \int_{-a}^{a} e^{-i2\pi yv} \left[\frac{e^{-i2\pi ux}}{-i2\pi u}\right]_{-(a^2-y^2)^{1/2}}^{(a^2-y^2)^{1/2}} dy$$

$$= \frac{K}{\pi u} \int_{-a}^{a} e^{-i2\pi vy} \sin\left(2\pi u (a^2-y^2)^{1/2}\right) dy$$

By symmetry,

$$= \frac{2K}{\pi u} \int_{0}^{a} \cos(2\pi vy)\, \sin\left(2\pi u (a^2-y^2)^{1/2}\right) dy$$

We can look this up and obtain

$$F(u,v) = K a (u^2+v^2)^{-1/2}\, J_1\left(2\pi a (u^2+v^2)^{1/2}\right)$$

$$= Ka \frac{J_1(2\pi a q)}{q} \qquad (q = \sqrt{u^2+v^2})$$

Messy, but possible. For a more complicated $f(r)$ it would have been impossible save for numerical methods.

If instead we use circular coordinates we can get a simpler result:

$$F(u,v) = \iint_{-\infty}^{\infty} f(r)\, e^{-i2\pi(ux+vy)}\, dx\, dy$$

$$= \int_0^{\infty}\int_0^{2\pi} f(r)\, e^{-i2\pi q r \cos(\theta-\phi)}\, r\, dr\, d\theta$$

The new kernel ($e^{-i2\pi q r \cos(\theta-\phi)}$) follows from expressing $ux+vy$ in polar coordinates. Letting (x,y) and (u,v) be 2-D vectors expressed in complex notation,

$$ux+vy = \text{Real part}\left[(x+iy)(u-iv)\right]$$

$$= \text{Real part}\left[re^{i\theta} q e^{-i\phi}\right]$$

$$= rq\cos(\theta-\phi)$$

Because we know $f(r)$ is independent of angle, we integrate out θ:

$$F(u,v) = \int_0^{\infty} f(r) \int_0^{2\pi} e^{-i2\pi q r \cos(\theta-\phi)}\, d\theta\, r\, dr$$

As before, since we integrate over all θ, we drop dependence on ϕ:

$$= \int_0^{\infty} f(r)\left[\int_0^{2\pi} \cos(2\pi q r \cos\theta)\, d\theta\right] r\, dr$$

where only the cosine part of the exponential is needed. (Sine part integrates to zero].

Recalling our integral definition for J_0:

$$J_0(r) = \frac{1}{2\pi}\int_0^{2\pi} \cos(r\cos\theta)\,d\theta$$

we obtain

$$F(u,v) = 2\pi \int_0^{\infty} f(r)\, J_0(2\pi q r)\, r\, dr$$

By symmetry, we can state the formula for the Hankel transform:

$$F(q) = 2\pi \int_0^{\infty} f(r)\, J_0(2\pi q r)\, r\, dr$$

The Hankel transform is a one-dimensional transform, but it is interpreted as a 2-D transform of a circularly symmetric function. We use these so often it is useful to note a series of Hankel transform pairs.

Disk or pillbox

We showed above that

$$\text{rect}\left(\frac{r}{2a}\right) \underset{H}{\supset} \frac{a\, J_1(2\pi a q)}{q}$$

This one shows up so often that it is useful to define the jinc function

$$\text{rect}(r) \underset{H}{\supset} \text{jinc}(q) = \frac{J_1(\pi q)}{2q}$$

Hence

$$\text{jinc } q = 2\pi \int_0^\infty \text{rect}(r)\, J_0(2\pi q r)\, r\, dr$$

Since the 2-D Fourier transform is reversible, so is the Hankel transform. Therefore, we can state

$$\text{rect}(r) = 2\pi \int_0^\infty \text{jinc } q\; J_0(2\pi q r)\, q\, dq$$

Ring Delta

If $f(r) = \delta(r-a)$, using the Hankel transform integral

$$F(q) = 2\pi \int_0^\infty f(r)\, J_0(2\pi q r)\, r\, dr$$

$$= 2\pi \int_0^\infty \delta(r-a)\, J_0(2\pi q r)\, r\, dr$$

$$= 2\pi\, a\, J_0(2\pi a q)$$

From the inverse formula,

$$\delta(r-a) = 2\pi \int_0^\infty 2\pi a\, J_0(2\pi a q)\, J_0(2\pi r q)\, q\, dq$$

This is the Bessel function orthogonality relation - Bessel functions of different "frequency" are orthogonal to each other. For large arguments, this is obvious from the approximation

$$J_0(\omega t) \approx \sqrt{\frac{2}{\pi \omega t}} \cos\left(\omega t - \frac{\pi}{4}\right)$$

From these relations, we see that the jinc function can be described as a circularly symmetric function containing all spatial frequencies up to some cutoff value, evenly weighted, and in all directions. Similarly, $J_0(r)$ contains only one frequency but at all orientations. There is thus a correspondence between

$$\text{sinc} \longleftrightarrow \text{jinc}$$
$$\text{cosine} \longleftrightarrow J_0$$

linear　　　　circularly symmetric

Similarity for Hankel transform

The similarity theorem for Hankel transforms follows from the two-D Fourier similarity theorem. The stretching (or compression) occurs equally in both dimensions x, y because of symmetry, hence

$$f(r) \underset{H}{\supset} F(q) \Rightarrow f\left(\frac{r}{a}\right) \underset{H}{\supset} a^2 F(aq)$$

We have used the symbol $\underset{H}{\supset}$ to designate Hankel transform, but we could also have simply used ${}^2\supset$. (Why?)

Hankel transform of annular slit

Another commonly encountered Hankel transform is that of the annular slit, which can be expressed

$$A \operatorname{rect}\left(\frac{r}{2a+w}\right) - A \operatorname{rect}\left(\frac{r}{2a-w}\right)$$

where A is the amplitude of the function over the slit. A picture of this function is:

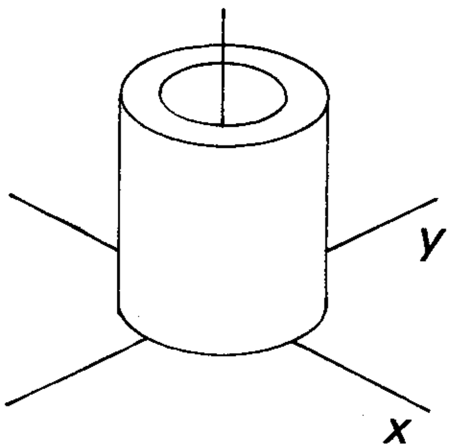


Figure 9-7 A circularly symmetrical function suitable for representing the distribution of light over a uniformly illuminated annular slit. By an extraordinary visual illusion connected with the evolution of vision, the height and outside diameter appear unequal.

If the width of the slit is w, as above, the impulse representation of the slit is

$$w A \delta(r-a)$$

and the equality between the representations should increase as $w \to 0$. Thus,

$$A(2a+w)^2 \operatorname{jinc}((2a+w)q) - A(2a-w)^2 \operatorname{jinc}((2a-w)q)$$

$$\underset{\substack{as \\ w\to 0}}{\longrightarrow} w A \, 2\pi a \, J_0(2\pi a q)$$

where the top line is the transform of the rect version of the slit and the bottom line is the δ-function version. Note that the upper line approaches a derivative:

$$\frac{\partial}{\partial a}\left(4a^2 \operatorname{jinc} 2qa\right) = 2\pi a \, J_0(2\pi a q)$$

which can be obtained from the derivative relations of Bessel functions.