<u>Sampling and Interpolation</u>

Sampling in 2-D again has many ~~much~~ more varieties than in 1-D. There are many coordinate systems - Cartesian, polar, for instance, but we can have line sampling as well. We have previously used the shah function, which was its own transform, for discussing point sampling.

Interpolation can be viewed as the inverse of sampling - it reconstructs a function from known sample values. The mapping goes from discrete to continuous. Very many manipulations of 2-D data require interpolation.

- Image enlargement
- Grey value generation
- Filtering
- Polar to rectangular conversion

Samples

Time dependent signals are often presented as a series of measurements at successive instances in time, and much computer processing of data relies on this interpretation. The spatial 2-D equivalence of this is data represented at uniform (or non-uniform) increments in space. Thus a 2-D image may be discussed as a matrix of individual measurements.

For any real data set, the number of points N is limited, as is the precision of the measurement. Often the precision is expressed as a number of "bits" per sample - values from 8 to 32 are common.

Samples are distinguished by assigning each an ordinal number that might range from 0 to $N-1$ or 1 to N, depending on whether you program in C or Fortran. 2-D data often can be described more easily by two numbers that are readily related to x and y positions, although the computer doesn't usually treat them this way.

δ-functions allow us to move freely back and forth between the continuous functions to be sampled and the discrete representation.

A-D converters. A-D converters are devices that input continuous signals and output discrete samples:

$$f(x) \longrightarrow \boxed{A/D} \equiv \text{digital data}$$

They often work with and integrate and dump front end, which calculates the average value of a function over an interval of time τ. Hence the output is

$$\text{out} = \frac{1}{\tau}\int_0^{\tau} f(x)\,dx \approx f(x) \text{ for short } \tau$$

Is this operation familiar to us? ~~tob~~ Write it as

$$out_{time\ x_0} = \frac{1}{\tau}\int_{-\infty}^{\infty} rect\left(\frac{x-x_0}{\tau}\right) f(x)\, dx$$

Higher speed A/D functions require shorter and shorter τ, leading to

$$out_{time\ x_0} = \int_{-\infty}^{\infty} \delta(x-x_0)\, f(x)\, dx$$

This might give you some idea physically how closely we need approach $\tau \to 0$ for a useful approximation.

Sampling at a point

Simply,

$$f(a,b) = \iint_{-\infty}^{\infty} f(x,y)\ {}^2\delta(x-a, y-b)\, dx\, dy$$

which we now know as an instance of the convolution

$$f(x,y) = \iint_{-\infty}^{\infty} f(x',y')\ {}^2\delta(x-x', y-y')\, dx'\, dy'$$

or $\quad f(x,y) = {}^2\delta(x,y) ** f(x,y)$

Thus viewed as a transfer function in system theory the sampling function ${}^2\delta(x,y)$ operates on $f(x,y)$, and doesn't change it. This convolution viewpoint will be of great use when we consider other sampling schemes.

Sampling by a pattern of points

Suppose we sample an image by an array of points and then combine groups of these points to produce a lower-resolution image. This is often the case when we scan a photograph on a high-density scanner and want to transmit the picture quickly over a limited bandwidth network. How can we describe the transfer function of such a system?

Model the operation as a set of N impulses at high-res, each of strength N^{-1}, at locations (a_i, b_i). Then the output of the system is

$$f(a_i, b_i) = \iint_{-\infty}^{\infty} f(x,y)\left[\frac{1}{N}\sum_{i=1}^{N} {}^2\delta(x-a_i, y-b_i)\right] dx\, dy$$

Recognize this as a convolution integral, with the input function $f(x,y)$ convolved with a pattern defined by (a_i, b_i).

Figure 7-1 A drawing (left) to be digitized at a coarse interval d, after which samples situated as on the right will be averaged in accordance with a given convolving pattern (lower left). The design is a medieval ruler-and-compass construction denoting purity. The radii are proportional to small integers, which facilitates layout and reproducibility. An interesting exercise is to attempt to improve the design by relaxing the pure digital tradition.

Defining the averaged function by $f_{av}(x,y)$

$$f_{av}(x,y) = \iint_{-\infty}^{\infty} f(x',y')\left[\frac{1}{N}\sum {}^2\delta(x-x'+a_i, y-y'+b_i)\right] dx'dy'$$

$$= \iint_{-\infty}^{\infty} f(x-x', y-y')\left[\frac{1}{N}\sum {}^2\delta(x'+a_i, y'+b_i)\right] dx'dy'$$

which is simply the convolution

$$f_{av}(x,y) = \left[\frac{1}{N}\sum {}^2\delta(x+a_i,\, y+b_i)\right] ** f(x,y)$$

Thus the <u>impulse response</u> of the system $h(x,y)$, which forms the convolving pattern, is

$$h(x,y) = \frac{1}{N}\sum {}^2\delta(x+a_i,\, y+b_i)$$

and the <u>transfer function</u> is $H(u,v) \supset h(x,y)$. In the example above,

$$h(x,y) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ \underline{1} & 1 & 1 \end{bmatrix}$$

where the underlined $\underline{1}$ gives the origin for reference. The convolution might be expressed now as

$$f_{av}(x,y) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ \underline{1} & 1 & 1 \end{bmatrix} ** f(x,y)$$

which neatly summarizes the sampling and smoothing nature of the sampling system. Examining the transfer function:

$$H(u,v) = 9 \operatorname{sinc} 3u \operatorname{sinc} 3v\, e^{-i2\pi(u+v)}$$

As u and $v \to 0$, the transfer function peaks at 9 and so amplifies the low frequency component, or, in other words, smooths the data. Nulls at $u = \frac{1}{3}$ or $v = \frac{1}{3}$ (signals with period 3) are filtered out completely, as $H(\frac{1}{3}, v) = H(u, \frac{1}{3}) = 0$.

This convolution viewpoint for sampling will prove a powerful tool for describing sampled data systems.

If the weights associated with the sampling pattern are nonuniform, then the more general expression below applies:

$$\frac{1}{N}\sum_{i=1}^{N} h_i \, {}^2\delta(x+a_i,\, y+b_i) \;\supset\; H(u,v)$$

Sampling along lines

Sometimes the sampling function we use occurs along lines rather than at discrete points, for example if we scan an image using a slit, or if we scan the sky with an antenna with a fan beam.

To express the sampled function, we can simply use the appropriate δ-function for the scanning aperture. For a unit-strength ~~[illegible]~~ impulse located on the $y=x$ line, for example

$$\iint_{-\infty}^{\infty} f(x,y)\,\delta(y-x)\,dx\,dy$$

which is simply the sifting property of the δ-function.

General Curvilinear sampling

Suppose we need to sample a function along a curve – again we can think of this problem in terms of our sifting theorem. For a curve defined by $c(x,y)=0$,

$$\iint_{-\infty}^{\infty} f(x,y)\,\delta\left[c(x,y)\right]dx\,dy = \int_c \alpha(x,y)\,f(x,y)\,ds$$

where s is the arc length along the curve and $\alpha(x,y)$ is the strength per unit length of the δ-function.

The Shah function and sampling

The shah function, we have stated previously, has two roles:

Sampling - obtained by multiplying a function by shah

Periodicity - obtained by convolving a function with shah

$$\text{Reminder: } {}^2\text{III}(x,y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} {}^2\delta(x-m, y-n)$$

Hence a sampled function at arbitrary spacing would be described by

$$\frac{1}{|XY|}\, {}^2\text{III}\left(\frac{x}{X}, \frac{y}{Y}\right) f(x,y)$$

This is a product in the time domain that we can represent by a convolution in the transform (frequency) domain.

Other sampling patterns follow from our previous discussions of the shah and its relatives, the various sampling patterns consisting of points, lines, and grilles.

In each case we can relate the sampled function $f_{samp}(x,y)$ to the transform of the convolved transforms of the original function and the sampling function:

$$f_{samp}(x,y) \overset{2}{\supset} F(u,v) ** H(u,v)$$

Factoring

We may use the above ideas to help decompose a complicated function into a series of simpler functions, each of which we can analyze. For example, a row of line impulses ${}^2\text{III}(y)$

can be viewed as the convolution of a series of δ's with a single line:

$$^2Ш(y) = \sum {}^2\delta(x, y-n) ** \delta(y)$$

$$= [Ш(y)\,\delta(x)] ** \delta(y)$$

Pictorially,

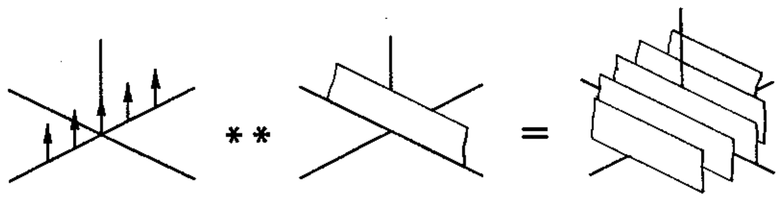

Figure 7-7 A way of generating the impulse grille by two-dimensional convolution between a row of impulses and a line impulse.

Another example: How about

$$\delta(x) ** \delta(y)$$

We can evaluate this directly using our 3-step rules:

$$\delta(x) ** \delta(y) = \iint \delta(x')\,\delta(y-y')\,dx'dy'$$

$$= \tau^{-2} \iint \mathrm{rect}\left(\frac{x'}{\tau}\right) \mathrm{rect}\left(\frac{y-y'}{\tau}\right) dx'\,dy'$$

$$= \tau^{-2} \int_{-\frac{\tau}{2}}^{\tau/2} dx' \int_{y'-\frac{\tau}{2}}^{y'+\tau/2} dy'$$

$$= 1$$

Note that we could have gotten this result directly from the convolution theorem:

$$\delta(x) ** \delta(y) \overset{2}{\supset} \delta(v) \cdot \delta(u) = {}^{2}\delta(u,v)$$

and $\cancel{3}$ ${}^{2}\delta(u,v) \overset{2}{\supset} 1$

hence ${}^{2}\delta(x) ** {}^{2}\delta(y) \overset{2}{\supset} 1$

The Two-Dimensional Sampling Theorem

"Given a continuous function $f(x,y)$ known at a set of grid points (x_i, y_i), intermediate values of the function between the grid points can be recovered exactly provided that $f(x,y)$ is band-limited"

Band-limited here is used in the sense analogous to its usage in radio bands, where a frequency band determined by a maximum and minimum frequency is defined. A band-limited signal $s(t)$ would have a transform $S(f)$ such that

$$S(f) = 0 \quad \text{if} \quad f < f_{min} \quad \text{or} \quad f > f_{max}$$

Examples:

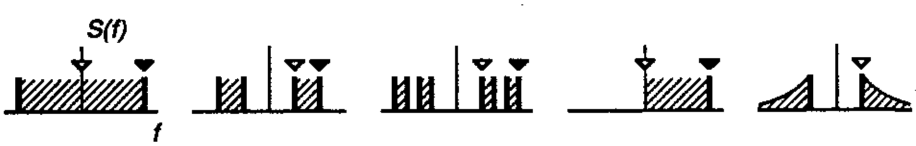

Figure 7-8 Some band arangements meeting the condition $S(f) = 0$ wherever $|f| > f_{\max}$ or $|f| < f_{\min}$ and one (on the right) that does not.

In 2-D, if the nonzero region of a Fourier transform $F(u,v)$ is restricted according to

$$F(u,v) = 0 \quad \text{for } |u| > u_{max},\ |v| > v_{max}$$

Then it is band-limited. Note that we have implicitly assumed that there is no low-frequency cutoff in the spatial frequency. Of course there might be, and we need to know this to calculate the bandwidth of the signal. But the condition must still be met if we assume it is band-limited.

Example. ${}^2\mathrm{sinc}\,(x \cdot 2u_{max}, y \cdot 2v_{max})$ is band-limited because its transform,

$$\frac{1}{|4u_{max}v_{max}|}\,{}^2\mathrm{rect}\left(\frac{u}{2u_{max}}, \frac{v}{2v_{max}}\right)$$

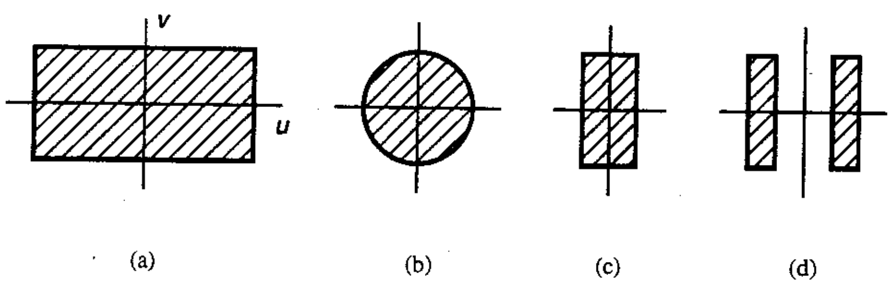

Figure 7-9 If the nonzero values of a transform $F(u, v)$ are confined to a finite area, as in the examples shown, the function $f(x, y)$ is band-limited.

Suppose the sampling grid is at unit locations in both dimensions:

$$s(x,y) = {}^2\Pi\!\Pi(x,y)\, f(x,y) = \sum\sum f(i,j)\, {}^2\delta(x-i, y-j)$$

where $s(x,y)$ represents the sampled version of $f(x,y)$.

Then the transform $S(u,v)$ may be written, using the convolution theorem,

$$S(u,v) = {}^2\text{III}(u,v) ** F(u,v)$$

How to picture this? Consider a band-limited function $f(x,y)$ with frequencies constrained to a small square near the origin.

The sampling function ${}^2\text{III}(x,y)$ has transform ${}^2\text{III}(u,v)$, where we have assumed unit spacing in (u,v) is larger than the square defined by the transform $F(u,v)$.

The convolution of the two spectra replicates the spectrum $F(u,v)$ periodically.

The inverse transform yields the sampled signal $s(x,y)$.

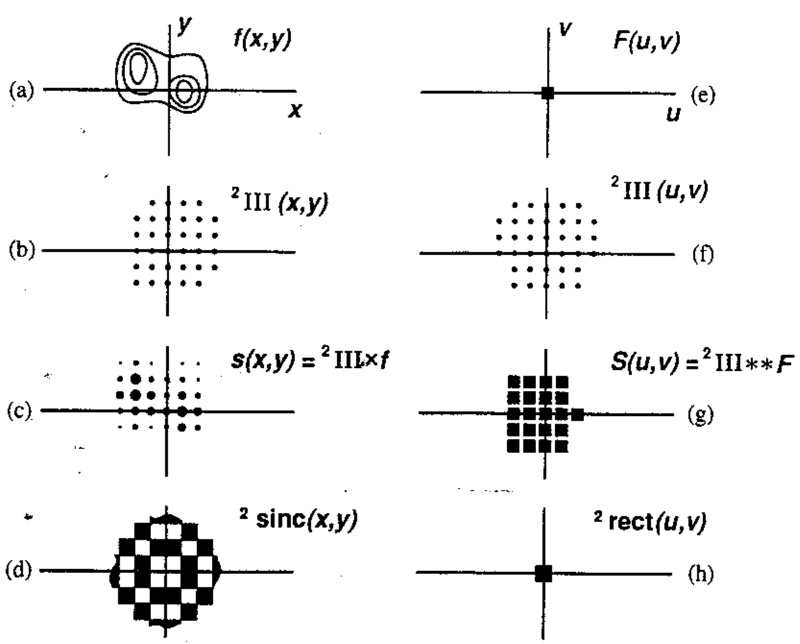

Figure 7-10 (a) A function $f(x,y)$ and its Fourier transform $F(u,v)$, (b) The sampling function ${}^2\text{III}(x,y)$ and its Fourier transform ${}^2\text{III}(u,v)$, (c) The sampled function ${}^2\text{sinc}(x,y)$ and its transform ${}^2\text{rect}(u,v)$, (d) The interpolating function ${}^2\text{sinc}(x,y)$ and its transform ${}^2\text{rect}(u,v)$.

Does this help understand the sampling theorem? Consider whether or not it is possible to recover $f(x,y)$ from $s(x,y)$. It is <u>if</u> we can isolate the single the proper repeated island in the transform domain, which we can clearly do if the island spacing is less than the bandwidth.

Thus, a fairly abstract notion of sampling adequacy is seen to be a trivial exercise when viewed as a transform manipulation. Such is the power of convolution and transforms!