$$\left(f(x,y) \star\star f(x,y) = f(x,y) ** f(-x,-y) \right)$$

Given autocorrelation theorem:

$$f(x,y) \star\star f^{*}(x,y) \supset F^{*}(u,v)\, F(u,v)$$

Complex version of autocorrelation is cross-correlation of a signal with its complex conjugate:

$$f(x,y) \star\star f^{*}(x,y) = f(x,y) ** f^{*}(-x,-y)$$

$$\Rightarrow \quad f(x,y) \star\star f^{*}(x,y) \supset F^{*}(u,v) \cdot F(u,v)$$

$$\iint f(x',y')\, f^{*}(x'+x, y'+y)\, dx'\, dy' \supset F^{*}(u,v) \cdot F(u,v)$$

$$= \iint_{-\infty}^{\infty} F^{*}(u,v)\, F(u,v)\, e^{-i2\pi(ux+vy)}\, du\, dv$$

if $x = y = 0$,

$$\iint |f(x,y)|^2\, dx\, dy = \iint_{-\infty}^{\infty} |F(u,v)|^2\, du\, dv$$

$\Rightarrow$ Rayleigh's theorem is an example of central value of transform

$f(0,0) = \iint F(u,v)\, du\, dv$

$[\text{ACF of } f] \supset |F|^2$

$(\text{ACF of } f)(0,0) = \iint |F|^2\, du\, dv$

$= \iint f f^{*}\, dx\, dy$

Proof of convolution theorem

Theorem: ~~Fe~~ If $h(x) = f(x) * g(x)$, $H(s) = F(s)\, G(s)$

Proof:
$$h(x) = f(x) * g(x)$$
$$= \int_{-\infty}^{\infty} f(x')\, g(x-x')\, dx'$$

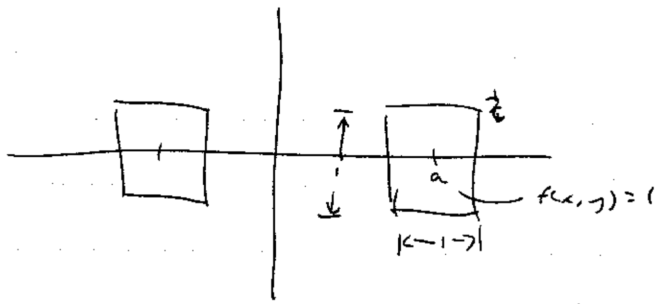
$$H(s) = \int_{-\infty}^{\infty} f(x')\, G(s)\, e^{-i2\pi x's}\, dx'$$

$$= \int_{-\infty}^{\infty} f(x')\, e^{-i2\pi x's}\, dx' \cdot G(s)$$

$$= F(s)\, G(s)$$

examples

$f(x,y)$

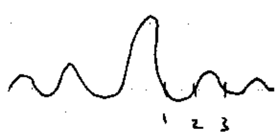


$$f(x,y) = \text{rect}(x,y) ** [\delta(x-a,y) + \delta(x+a,y)]$$

$$F(u,v) = \text{sinc}(u,v) 2\cos \pi a u$$

$\cancel{f(x,y)}$ $H(u,v) = F(u,v)\, G(u,v)$

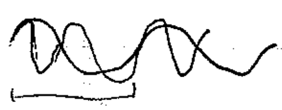
$$f(x,y) = \text{rect}(x,y) \Rightarrow F(u,v) = \text{sinc}(u,v)$$



$\cancel{G(u,v)} =$

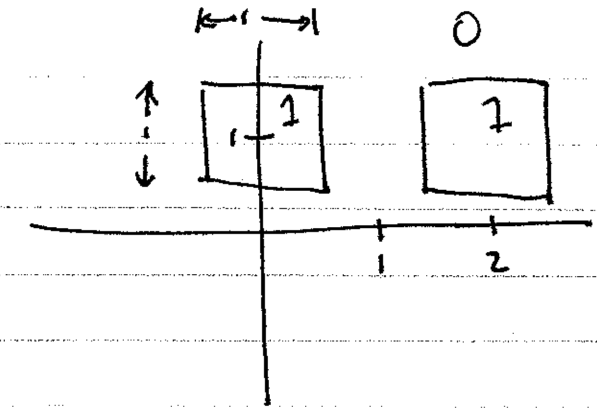
$$g(x,y) = a_1 \cos \pi x + a_2 \cos 2\pi x + a_3 \cos 3\pi x \cdots$$

$f(x,y) = \text{rect}(x,y)$



let $g(x,y) = \cos \pi x$

Example of computing transforms



Method 1: Have two rects, one shifted by (0, 1), the other by (2, 1)

First rect: transform is $\operatorname{sinc}(u,v)\, e^{-i2\pi v}$

Second: $\operatorname{sinc}(u,v)\, e^{-i2\pi(2u+v)}$

$$F(u,v) = \operatorname{sinc}(u,v)\left[e^{-i2\pi v} + e^{-i2\pi(2u+v)}\right]$$

$$= \operatorname{sinc}(u,v)\left[e^{-i2\pi v} e^{+i2\pi u} + e^{-i2\pi v} e^{-i2\pi u}\right] e^{-i2\pi u}$$

$$= \operatorname{sinc}(u,v)\, e^{-i2\pi v} e^{-i2\pi u}\left(e^{i2\pi u} + e^{-i2\pi u}\right)$$

$$= \operatorname{sinc}(u,v)\, e^{-i2\pi(u+v)} \cdot 2\cos 2\pi u$$

Method 2: One rect, convolved with two δ's, shifted

$\square \ast\ast \uparrow \ \uparrow$ and shifted up and right

$$\operatorname{sinc}(u,v) \cdot 2\cos 2\pi u \ e^{-i2\pi(u+v)}$$