## Central Value and Volume of Autocorrelation

Two useful theorems when working with autocorrelation are the central value and volume theorems, which we often use to check calculations and summarize properties.

For an autocorrelation $c(x,y)$, the central value is found by evaluating at $0,0$:

$$c(0,0) = \iint_{-\infty}^{\infty} (f(x,y))^2 \, dx\, dy$$

As for volume,

$$\iint_{-\infty}^{\infty} c(x,y)\, dx\, dy = \left[ \iint_{-\infty}^{\infty} f(x,y)\, dx\, dy \right]^2$$

So we see that the central value is the volume under the square of the original function. The analog in one-d is that the value of a time series autocorrelation at 0 displacement is the power in the original signal.

The volume under the autocorrelation is the square of the volume under the original function.

How do these relate to the chat function? The volume of the pillbox is height × area, and height $=1$ and area $= \pi r^2 = \frac{\pi}{4}$. Thus the square of the pillbox height is also 1, yielding

$$\text{chat}(0) = \frac{\pi}{4}$$

$$\text{volume}(\text{chat}) = \frac{\pi^2}{16}$$

We can derive the volume theorem as follows.

$$\iint c(x,y)\,dx\,dy = \iint \left[ \iint f(x'-x, y'-y)\, f(x',y')\,dx'\,dy' \right] dx\,dy$$

We can rearrange the order of integration as follows if the integral of $f$ exists, which we'll assume:

$$\text{volume} = \iint f(x',y') \left[ \iint f(x'-x, y'-y)\,dx\,dy \right] dx'\,dy'$$

$$= \iint f(x',y') \left[ \iint f(x,y)\,dx\,dy \right] dx'\,dy'$$

$$= \iint f(x',y')\,dx'\,dy' \iint f(x,y)\,dx\,dy$$

$$= \left[ \iint f(x,y)\,dx\,dy \right]^2$$

The autocorrelation volume theorem is a special case of an area theorem for convolution, which in 1-D is

$$\int_{-\infty}^{\infty} h(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx \int_{-\infty}^{\infty} g(x)\,dx \quad \text{if} \quad h = f * g$$

Again the 2-D case is obvious.

## Convolution Sum

Numerical evaluation of convolution requires the following sum:

$$h_{ij} = \sum_k \sum_l f_{i-k,\,j-l}\, g_{k,l}$$

where the $f$, $g$, and $h$ coefficients can be taken to be samples of continuous processes. If greater accuracy is needed the sampling interval can be adjusted as needed, and doubling the number of points in both dimensions leads to a fourfold increase in the number of computations needed.

Example sums

Consider the following case:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} ** \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

Invert f and visualize

$$\begin{bmatrix} 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

Shifting and multiplying and adding gives

$$\begin{bmatrix} 1 & 2 & 3 & \cdots \\ 5 & 8 & 12 & \cdots \\ 6 & 14 & 24 & \cdots \\ 8 & 14 & 21 & \cdots \end{bmatrix}$$ ← different than in book!

As a second example, consider the self-convolution:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} ** \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 2 & 2 & 2 & 0 & 0 \\ 3 & 4 & 6 & 4 & 2 \\ 2 & 2 & 4 & 2 & 2 \\ 3 & 6 & 11 & 6 & 3 \\ 2 & 2 & 4 & 2 & 2 \\ 2 & 4 & 6 & 4 & 3 \\ 0 & 0 & 2 & 2 & 2 \\ 1 & 2 & 3 & 2 & 1 \end{bmatrix}$$

What can be say about areas and volumes? Our theorem was

$$\iint h(x,y)\,dx\,dy = \iint f(x,y)\,dx\,dy \iint g(x,y)\,dx\,dy$$

~~Thus~~ The sum of the terms of the original matrix is 11, and the sum of the larger matrix is $121 = 11^2$, so it checks.

## Computing convolutions

Convolving two matrices involves two input matrices $f$ and $g$ with $M_f$ and $N_f$ columns and rows, and $M_g$ and $N_g$ columns and rows. The resultant $h$ matrix will be of size $M_f + M_g - 1 \times N_f + N_g - 1$. While fast transform methods greatly speed up computations and are validated by the convolution theorem, it is simple but tedious to evaluate the convolution by computer. Bracewell gives a simple program to do this:

CONVOLUTION OF $f(\,,\,)$ WITH $g(\,,\,)$

```
FOR i=1 TO Nf+Ng-1
    FOR j=1 TO Mf+Mg-1
       h(i,j)=0
       FOR k=MAX(1,i+1-Ng) TO MIN(i,Ng)
          FOR l=MAX(1,j+1-Mg) TO MIN(j,Mg)
             P=f(k,l)
             Q=g(i+1-k,j+1-l)
             h(i,j)=h(i,j)+P*Q
          NEXT l
       NEXT k
    NEXT j
NEXT i
```

## Example application – smoothing

One common application of the above is for digital smoothing of data. Here we convolve a data matrix $g$ with a usually smaller matrix $f$, for example, for a simple boxcar smoothing we might use

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} ** \; g$$

and we might normalize the entire result by $\frac{1}{9}$ so the volume under $g$ is unchanged. Choosing the proper kernel $f$ depends on a detailed analysis of its spectral properties, which we'll address after we discuss the 2-D convolution theorem.

## Matrix-product notation

We can express the convolution ideas shown above in terms of matrix products by proper manipulation of the ~~2D~~ data. One example is given in the text and we'll reproduce it here to show how it can be done. It is, however, rather tedious to implement and usually we use explicit codes such as the example at the top of this page.

$$[f]=\begin{bmatrix}0&0&0&0\\0&1&2&0\\0&3&4&0\\0&0&0&0\end{bmatrix},\qquad [g]=\begin{bmatrix}0&1&2&0\\1&7&9&2\\3&11&13&4\\0&3&4&0\end{bmatrix},$$

$$[\mathbf{f}]=\begin{bmatrix}0\\0\\0\\0\\0\\1\\2\\0\\0\\3\\4\\0\\0\\0\\0\\0\end{bmatrix},\qquad [\mathbf{g}]=\begin{bmatrix}0\\1\\2\\0\\1\\7\\9\\2\\3\\11\\13\\4\\0\\3\\4\\0\end{bmatrix}.$$

**Figure 5-24** An object $[f]$ and image $[g]$ represented by matrices, and the corresponding column vectors $[\mathbf{f}]$ and $[\mathbf{g}]$.

$$\begin{bmatrix}
2&1&0&0&1&0&&&&&&&1&0&0&1\\
1&2&1&0&0&1&0&&&&&&&1&0&0\\
0&1&2&1&0&0&1&0&&&&&&&1&0\\
0&0&1&2&1&0&0&1&0&&&&&&&1\\
1&0&0&1&2&1&0&0&1&0&&&&&&\\
&1&0&0&1&2&1&0&0&1&0&&&&&\\
&&1&0&0&1&2&1&0&0&1&0&&&&\\
&&&1&0&0&1&2&1&0&0&1&0&&&\\
&&&&1&0&0&1&2&1&0&0&1&0&&\\
&&&&&1&0&0&1&2&1&0&0&1&0&\\
&&&&&&1&0&0&1&2&1&0&0&1&0\\
0&&&&&&&1&0&0&1&2&1&0&0&1\\
1&0&&&&&&&1&0&0&1&2&1&0&0\\
0&1&0&&&&&&&1&0&0&1&2&1&0\\
0&0&1&0&&&&&&&1&0&0&1&2&1\\
1&0&0&1&0&&&&&&&1&0&0&1&2
\end{bmatrix}
\begin{bmatrix}0\\0\\0\\0\\0\\1\\2\\0\\0\\3\\4\\0\\0\\0\\0\\0\end{bmatrix}
=
\begin{bmatrix}0\\1\\2\\0\\1\\7\\9\\2\\3\\11\\13\\4\\0\\3\\4\\0\end{bmatrix}$$

The argument from linearity tells us that $[\mathbf{f}]$ and $[\mathbf{g}]$, if regarded as simple sequences rather than as formal column vectors in matrix terminology, must be related by one-dimensional convolution; in other words $\{g\}=\{h\}*\{f\}$, and in this case

$$\{h\}=\{0100\ 1210\ 0100\ 0000\}.$$

$$g(x,y)=h(x,y)**f(x,y)=\begin{bmatrix}0&1&2&0\\1&7&9&2\\3&11&13&4\\0&3&4&0\end{bmatrix}$$

$$=\begin{bmatrix}0&1&0&0\\1&2&1&0\\0&1&0&0\\0&0&0&0\end{bmatrix}**\begin{bmatrix}0&0&0&0\\0&1&2&0\\0&3&4&0\\0&0&0&0\end{bmatrix}.$$

## The Two-Dimensional Convolution Theorem

This material comes from Chapter 6 in the text. Please read.

In one-d the convolution theorem appears whenever we have systems exhibiting both linearity and time invariance. The equivalent in 2-D is linearity and space invariance, or shift invariance. What this means is that the impulse response of the system doesn't change much as the location on the x,y plane changes.

Space invariance is strictly very hard to come by, because most systems have some distortions that grow more pronounced with distance from the center. But if the distortion is not too great, convolution applies to some degree and we can derive an image distribution from a source distribution according to

$$i(x,y) = h(x,y) ** s(x,y)$$

where $h(x,y)$ represents the impulse response, or optical transfer function, of the imaging system. For example,

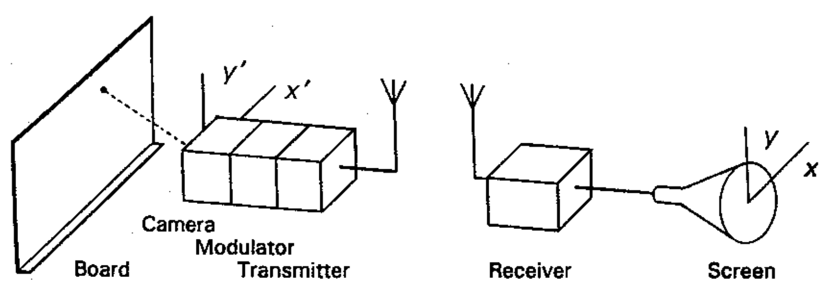



**Figure 6-1** Given linearity and space invariance, the image $i(x, y)$ on the screen is derivable from the object distribution $o(x, y)$ by two-dimensional convolution with the point response function $h(x, y)$ of the system.

Here $h(x,y)$ represents the total transfer function from source to image, and would have components due to each block shown.

## Statement of the Theorem

The convolution theorem looks pretty much the same in either 1-D or 2-D. In words "Convolution in one domain corresponds to multiplication in the transform domain". In one-d one of the functions is time-reversed prior to multiplication and addition, in 2-D one is rotated 180° first.

1-D Convolution theorem:

If $f(x) \supset F(s)$ and $g(x) \supset G(s)$,

$$f(x) * g(x) \supset F(s)\, G(s)$$

2-D convolution theorem:

If $f(x,y) \overset{2}{\supset} F(u,v)$ and $g(x,y) \overset{2}{\supset} G(u,v)$

$$f(x,y) ** g(x,y) \overset{2}{\supset} F(u,v)\, G(u,v)$$

Pictorially,

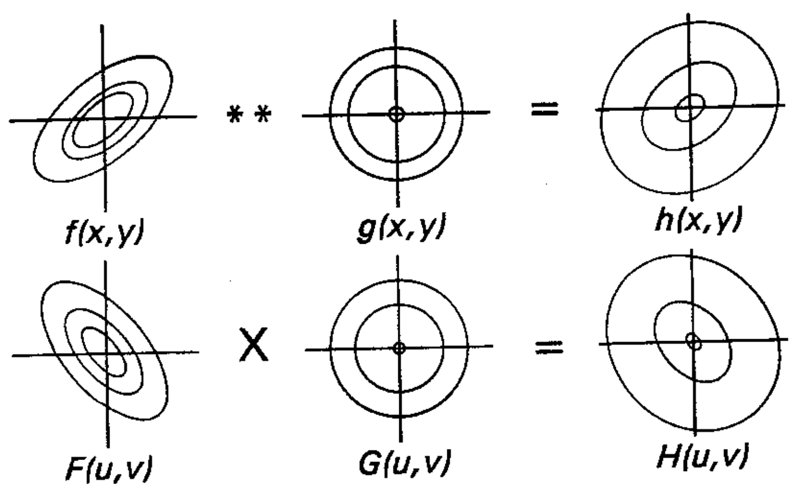



**Figure 6-2** Functions $f(x, y)$ and $g(x, y)$ are convolved to give $h(x, y)$, while their Fourier transforms, shown below, combine through multiplication.

The inverse form:

$$f(x,y)\, g(x,y) \overset{2}{\supset} F(u,v) ** G(u,v)$$

## Instrumental Effects

For convolution to hold, only translations of the scanning function are allowed. In some instrumentation rotation of the beam is inherent with translation, for example certain telescope mounts. Unless the scanning function has circular symmetry the image distribution is not described by convolution.

## Point response and transfer function

Notation, terminology, and the convolution theorem:

$$i(x,y) = \underset{\substack{\uparrow \\ \text{Point} \\ \text{response}}}{h(x,y)} ** s(x,y)$$

$$I(u,v) = \underset{\substack{\uparrow \\ \text{Transfer} \\ \text{function}}}{H(u,v)}\, S(u,v)$$

Sometimes we encounter the term modulation transfer function: $|H(u,v)|$ is the same as the magnitude of $H(u,v)$.

## The Autocorrelation Theorem

Autocorrelation is easily related to convolution through:

$$f(x,y) \star\star f(x,y) = f(x,y) ** f(-x,-y)$$

Since $f(x,y) \overset{2}{\supset} F(u,v)$, and $f(-x,-y) \overset{2}{\supset} F(-u,-v)$ (why?)

If $f(x,y)$ is real, then $F(-u,-v) = F^*(u,v)$

$$f(x,y) \star\star f(x,y) \overset{2}{\supset} F^*(u,v)\,F(u,v) = |F(u,v)|^2$$

In the complex $f(x,y)$ case,

$$f(x,y) \star\star f(x,y) = f(x,y) ** f^*(-x,-y)$$

That the autocorrelation is fundamental can be seen if we note that we can derive Rayleigh's theorem from it. From the autocorrelation theorem, we have

$$\iint_{-\infty}^{\infty} f(x',y')\, f^*(x'+x, y'+y)\, dx'\, dy' = \iint_{-\infty}^{\infty} |F(u,v)|^2 e^{i2\pi(ux+vy)}\, du\, dv$$

and if $x = y = 0$, we obtain

$$\iint_{-\infty}^{\infty} f(x',y')\, f^*(x',y')\, dx'\, dy' = \iint_{-\infty}^{\infty} |F(u,v)|^2\, du\, dv$$

which is Rayleigh's theorem directly. The theorem can now be understood as the property that the central value of an autocorrelation is equal to the integral of the transform, or, actually, for any function:

$$\text{if } f(x,y) \supset F(u,v),\ f(0,0) = \iint F(u,v)\, du\, dv$$

Application: evaluate

$$\iint_{-\infty}^{\infty} \operatorname{sinc} ax \operatorname{sinc} by\, dx\, dy$$

Since $\operatorname{sinc} ax \supset \frac{1}{|a|} \operatorname{rect} \frac{x}{a}$ and $\operatorname{sinc} by \supset \frac{1}{|b|} \operatorname{rect}\left(\frac{x}{b}\right)$ the integral

reduces to $\frac{1}{|a|} \operatorname{rect} 0 \ \frac{1}{|b|} \operatorname{rect} 0 = \frac{1}{|ab|}$

## Cross-correlation Theorem

If $f(x,y) \supset F(u,v)$ and $g(x,y) \supset G(u,v)$

Then $f(x,y) \star\star\, g(x,y) \supset F^*(u,v)\, G(u,v)$

Note that the order matters, it determines which function gets the conjugate asterisk.

## Factorization and Separation

If we are to transform a function, it often helps to factor the function and use the convolution theorem to simplify the procedure. Where the factors are all one-dimensional the function is separable also, and transformation may be quite simple.

For example, consider the matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

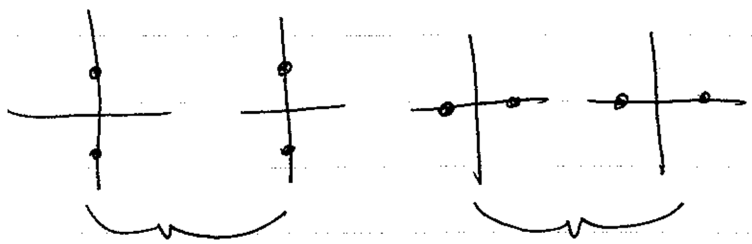
What is its transform $F(u,v)$? Although not obvious, we can break the problem down to

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} ** \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$

The transform of the convolution will be the product of the individual transforms of $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $[1 \;\; 2 \;\; 1]$. Bracewell shows (p. 211) that since $[1 \;\; 2 \;\; 1] = \left[{}^2\delta(x+1,y) + 2\,{}^2\delta(x,y) + {}^2\delta(x-1,y)\right]$

$= \left[{}^2\delta(x+\tfrac{1}{2},y) + {}^2\delta(x-\tfrac{1}{2},y)\right] ** \left[{}^2\delta(x+\tfrac{1}{2},y) + {}^2\delta(x-\tfrac{1}{2},y)\right]$

The array represents the fourfold convolution of these factors:



$\cos^2 \pi v$ $\cos^2 \pi u$

and the total transform is $16 \cos^2 \pi v \cos^2 \pi u \, \mathrm{sinc}\, v \, \mathrm{sinc}\, u$.

## Comptational Complexity

Convolution implemented in the simplest convolution sum manner requires a number of computer operations proportional to $N^4$, because each array is of size $N \times N$ and $N \times N$ locations must be evaluated. Therefore transform methods, which require $N^2 (\log N)^2$ operations for an $N \times N$ matrix give a considerable speedup for large $N$.

Thus most signal processing today that is convolutional in nature is implemented using FFT techniques. This imposes cyclic properties on the results and also requires that the data be adequately sampled to avoid aliasing, but the speedup makes feasible many more applications than would be possible elsewise.

To illustrate, for a $10^8$ Mflop machine, and assuming complex data, how long does a 4K × 4K 2-D ~~convol~~ convolution require?

Simple method: $N^4 / 10^8 \, \cancel{M}\text{flop} = \dfrac{4096^4}{10^8}$ seconds $= 2.8 \times 10^6$ seconds $\approx 33$ days

Transform method: $\dfrac{N^2 (\log_2 N)^2}{10^8 \, \cancel{M}\text{flop}} = \dfrac{4096^2 \times 12^2}{10^8 \, \text{flop}} = 24$ seconds