EE262 HANDOUT # 13

The Shift Theorem and Discrete Fourier Transform

We have examined the shift theorem

$$f(x,y) \supset F(u,v) \Rightarrow f(x-a, y-b) \supset F(u,v)\, e^{-i2\pi(au+bv)}$$

regarding its application to interpolation. Let's consider what this means in practice, where we use FFTs, a discrete form of the Fourier transform.

Let's look at the 1-D case as an example; the 2-D case is quite similar. We start with

$$f(x-a) \supset F(s)\, e^{-i2\pi as}$$

So, for a unit shift in position, $(a=1)$, the phase added to each coefficient in the frequency domain is proportional to s, with 2π phase added at $s=1$.

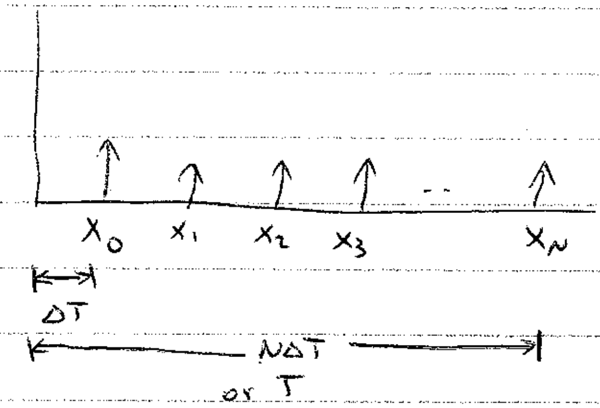
Properties of the FFT:

Sequence: $x_0, x_1, x_2 \cdots x_N$
Transform: $s_0, s_1, s_2, \cdots s_N$

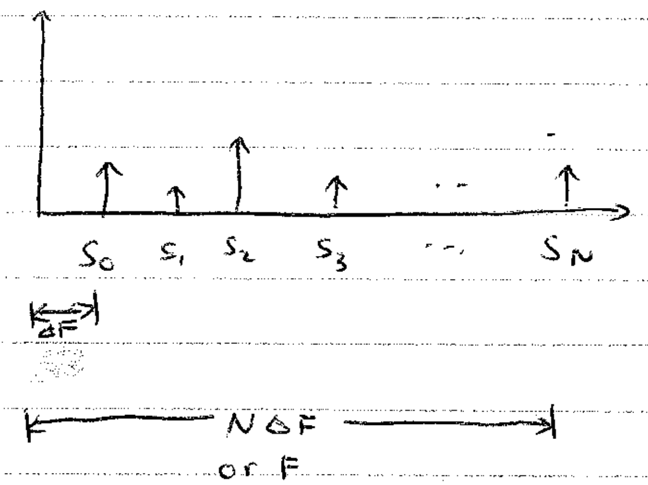
Let's say the sequence is T seconds long, then the time between samples is $\frac{T}{N} = \Delta T$.

The maximum frequency in the transform dimension is $F = \frac{1}{\Delta T}$, although we often think of this as $\pm \frac{F}{2}$. The frequency spacing is then $\Delta F = \frac{F}{N} = \frac{1}{N\Delta T}$

Time domain:



Frequency domain:



Note: If $\Delta T = 1$, then $F = 1$, so that applying a phase gradient of 2π over the entire frequency range shifts the original sequence by 1 point. In this instance the phase increment per frequency bin is $\frac{2\pi}{N}$.

So, for the practical case, to shift a sequence by a bins, we would apply a gradient to the spectrum $F(s)$ such that

$$F(s)' = F(s)\, e^{-i 2\pi a s}$$

If the original sequence of coefficients s_j is

$s_0, s_1, s_2 \cdots s_N$

Then

$$s_j' = s_j e^{-i 2\pi a \left(\frac{j}{N}\right)} \qquad j = 0, \frac{N}{2}-1$$

$$= s_j e^{-i 2\pi a \left(\frac{j-N-1}{N}\right)} \qquad j = \frac{N}{2}, N$$

So for each pixel of desired shift, the total phase across the spectrum is 2π radians.

On evaluating 2-D transforms

Our expression for the 2-D transform is

$$F(u,v) = \iint_{-\infty}^{\infty} f(x,y)\, e^{-i 2\pi (ux + vy)}\, dx\, dy$$

This implies kernels that vary in x and y, so that each coefficient requires a separate 2-D integration. But note we can rewrite the integral as

$$F(u,v) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x,y)\, e^{-i 2\pi ux}\, dx \right] e^{-i 2\pi vy}\, dy$$

The inner integral is a one-dimensional transform of $f(x,y)$, done "line by line". So we have, after evaluation of the inner integral,

$$F(u,v) = \int_{-\infty}^{\infty} f'(u,y)\, e^{-i 2\pi vy}\, dy$$

a second one-d transform but in the orthogonal direction. We have let $f'(u,y)$ be the transform in the x-direction of $f(x,y)$.

Thus, a 2-D transform can be computed using 2 sets of 1-D transforms, greatly reducing computational requirements. Each direction is done using repeated row/column transforms, as: