Theorems for the Two-dimensional Fourier Transform

Similarity Theorem

Note that in 2-D the similarity theorem only relates the scaling in the x and y dimensions, and not in an arbitrary direction. For example, stretching in the 45° direction cannot be represented by stretching in x and y only - this requires use of a rotation and stretch.

Similarity in one-d:

$$f(x) \supset F(s) \Rightarrow f(ax) \supset \frac{1}{|a|} F\left(\frac{s}{a}\right)$$

Simlarity in 2-D:

$$f(x,y) \overset{2}{\supset} F(u,v) \Rightarrow f(ax,by) \overset{2}{\supset} \frac{1}{|ab|} F\left(\frac{u}{a}, \frac{v}{b}\right)$$

Shift theorem

Shifting a point in one-d does not alter the magnitude of a function or of its Fourier components, but the phase of each component changes. In fact, each component $\cos 2\pi ft$ shifts in phase proportional to its frequency. Thus the shift in time leads to a linear phase gradient in the transform domain.

Shift in 1-D:

$$f(x) \supset F(s) \Rightarrow f(x-a) \supset e^{-i2\pi as} F(s)$$

Note that the phase shift at zero frequency is zero.

Shift in 2-D:

$$f(x,y) \overset{2}{\supset} F(u,v) \Rightarrow f(x-a,y-b) \overset{2}{\supset} e^{-i2\pi(au+bv)} F(u,v)$$

Once again, no change in amplitude occurs but a plane phase gradient through the origin is introduced:

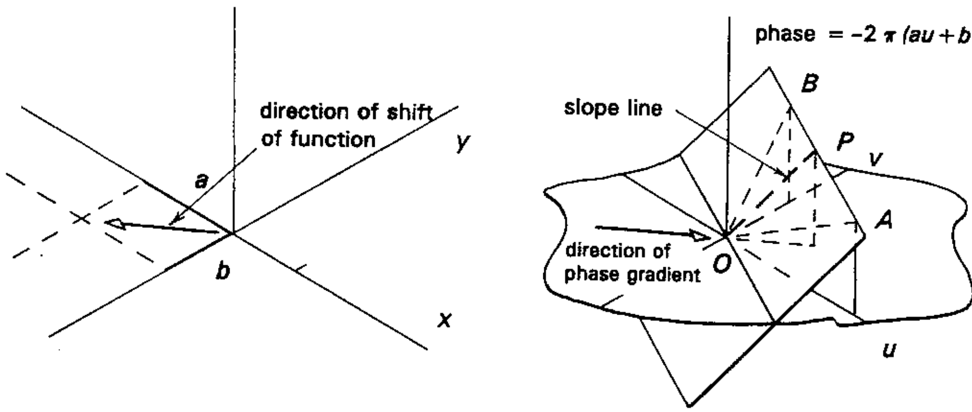


Figure 4-16 A shift of origin in the picture plane introduces a linear phase gradient in the transform domain.

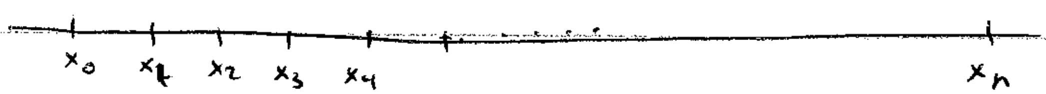
The slope of this plane is $2\pi\sqrt{a^2+b^2}$ with respect to the uv plane, equal to the gradient of the linear phase term.

$$\left[\Phi = -2\pi(au+bv) \quad , \text{grad } \Phi = \frac{\partial \Phi}{\partial u}\hat{u} + \frac{\partial \Phi}{\partial v}\hat{v} \right]$$

The converse of this theorem states that a phase gradient applied to an image, going through zero at the origin, results in a shift in the transform domain. This is useful for interpolation.

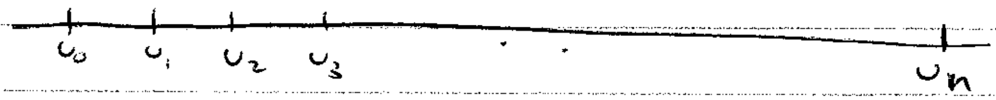
Example. How might we use the shift theorem to interpolate a sequence 1/4 pixel?

Consider this sequence of points sampled at each integer



which we represent by $f(x)$ at integral x.

We might calculate it transform, defined at coefficients u_j.



Now apply a phase gradient such that $\frac{\pi}{2}$ radians is added at each points:

$$u_j' = u_j e^{-2\pi i \frac{\pi}{2} \cdot j} \quad \text{or} \quad F'(u) = F(u)\, e^{-i \frac{\pi}{2} u} = F(u) e^{-i 2\pi \frac{u}{4}}$$

Inverse transform, and obtain

$$F(u)\, e^{-i 2\pi u \cdot \frac{1}{4}} \supset f(x - \tfrac{1}{4})$$

which, if evaluated at integral locations has values offset by $\frac{1}{4}$ point from the original sequence.

<u>Rotation theorem</u>

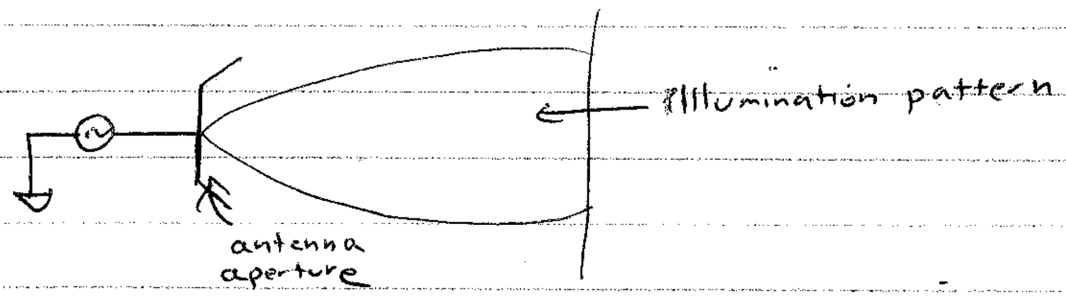
$$f(x,y) \overset{2}{\supset} F(u,v) \Rightarrow$$

$$f(x\cos\theta - y\sin\theta,\ y\cos\theta + x\sin\theta) \overset{2}{\supset} F(u\cos\theta - v\sin\theta,\ v\cos\theta + u\sin\theta)$$

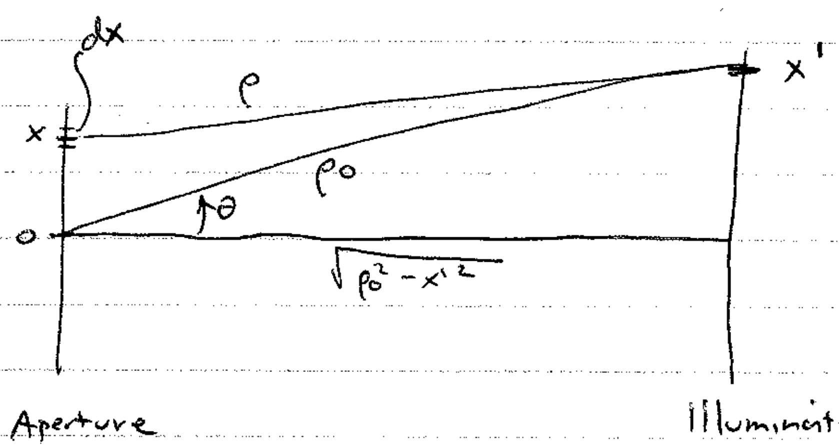
Here is a theorem that doesn't exist in 1-D analog. It also isn't exactly clear from the math. But if we consider the transform as a distibution of power from an antenna aperture, it is physically obvious that rotating an antenna yields a rotated pattern, θ for θ.

Example application. Fraunhofer approximation to antenna patterns

The Fraunhofer approximation relates the pattern, or distribution of energy, emitted by an antenna to the voltage distribution at the antenna aperture:



Physics: The voltage at a point x' on an illuminated surface is the sum of all the voltages for each infinitesimal element of the aperture, each weighted by an amplitude and phase propagation constant $\dfrac{e^{i\frac{2\pi}{\lambda}\rho}}{\frac{2\pi}{\lambda}\rho}$



Thus the total voltage sensed at x' is

$$\int_{-\infty}^{\infty} f(x)\, \frac{e^{i\frac{2\pi}{\lambda}\rho}}{\frac{2\pi}{\lambda}\rho}\, dx$$

where $f(x)$ is the aperture distribution and ρ is a function of both x and x'.

Geometry gives us

$$\rho^2 = (x-x')^2 + (\rho_0^2 - x'^2)$$

$$= \rho_0^2 - \underline{2xx'} + x^2$$

The Fraunhofer approximation is for small x, so we ignore x^2:

$$\rho^2 \approx \rho_0^2 - 2xx'$$

$$\rho \approx \rho_0 \left(1 - \frac{2xx'}{\rho_0^2}\right)^{1/2}$$

and, since $xx' \ll \rho_0^2$,

$$\rho \approx \rho_0 \left(1 - \frac{xx'}{\rho_0^2}\right) = \rho_0 - \frac{xx'}{\rho_0}$$

Using $\sin\theta = \dfrac{x'}{\rho_0}$

$$\rho = \rho_0 - x\sin\theta$$

So our integral is

$$\int_{-\infty}^{\infty} f(x) \frac{e^{i\frac{2\pi}{\lambda}(\rho_0 - x\sin\theta)}}{\frac{2\pi}{\lambda}\rho}\, dx$$

Again the small x approximation means we can ignore the amplitude variation caused by the denominator, and the ρ_0 in the exp( ) leads only to a phase constant which we ignore:

$$g(\theta) = \int_{-\infty}^{\infty} f(x)\, e^{-i 2\pi x \frac{\sin\theta}{\lambda}}\, dx$$

The pattern $g(\theta)$ is thus the Fourier transform of the aperture distribution $f(x)$ with $\frac{\sin\theta}{\lambda}$ as the argument. For small angles $\sin\theta \approx \theta$, and λ provides a scaling factor. The corresponding development in 2-D yields

$$g(\theta, \phi) = \iint_{-\infty}^{\infty} f(x,y)\, e^{-i 2\pi \left(x \frac{\sin\theta}{\lambda} + y \frac{\sin\phi}{\lambda} \right)}\, dx\, dy$$

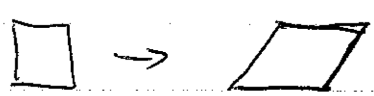
<u>Back to the theorems</u>

<u>Shear theorems.</u> Similarity, shift, and rotation are all special cases of the general affine transformation

$$x' = ax + by + c$$
$$y' = dx + ey + f$$

So is pure shear:

$$\left.\begin{aligned} x' &= x + by \\ y' &= y \end{aligned}\right\} \text{shear in } x \text{ (horizontal)}$$

 ⇐ Is this area-preserving?

<u>Simple shear:</u>

$$f(x,y) \overset{2}{\supset} F(u,v) \Rightarrow f(x+by, y) \overset{2}{\supset} F(u, v - bu)$$

So shear in one direction corresponds to another shear in the perpendicular direction for ~~the~~ the transformed data.

For completeness, vertical shear:

$$f(x, dx+y) \overset{2}{\supset} F(u-dv, v)$$

<u>Compound shear.</u>

$$f(x,y) \overset{2}{\supset} F(u,v) \Rightarrow$$

$$f(x+by, dx+y) \overset{2}{\supset} \frac{1}{|1-bd|} F\left(\frac{u-dv}{1-bd}, \frac{-bu+v}{1-bd}\right)$$

It is clear from the form of this pair that some scaling is involved. That ~~this~~ is so follows from a closer look at the transformations invoked.

Horizontal shear - $\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$

Vertical Shear - $\begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix}$

Now, for horizontal followed by vertical shear

$$\begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & b \\ d & 1+bd \end{bmatrix} \quad \Leftarrow \text{scaling in } y$$

Vertical followed by ~~hozn~~ horizontal

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix} = \begin{bmatrix} 1+bd & b \\ d & 1 \end{bmatrix} \quad \Leftarrow \text{scaling in } x$$

Thus the order of applying shear matters! Compound shear is different yet:

$$\begin{bmatrix} 1 & b \\ d & 1 \end{bmatrix} \quad \Leftarrow \text{This is the matrix we used for the theorem.}$$

Affine Theorem

The preceding have all been special cases of a general affine transformation. We may thus expect to be able to derive the case for the full transformation, and the answer is:

$$f(x,y) \overset{2}{\supset} F(u,v)$$

$$f(ax+by+c,\, dx+ey+f) \overset{2}{\supset}$$

$$\frac{1}{|\Delta|} \exp\left\{ \frac{i2\pi}{\Delta} \left[(ec-bf)u + (af-cd)v \right] \right\} F\left(\frac{eu-dv}{\Delta}, \frac{-bu+av}{\Delta} \right)$$

where Δ is the determinant $\begin{vmatrix} a & b \\ d & e \end{vmatrix} = ae - bd$

Bracewell gives a derivation in the book, pp 160-161.

Rayleigh's Theorem in 2-D

$$f(x,y) \overset{2}{\supset} F(u,v) \Rightarrow \iint |f(x,y)|^2 \,dx\,dy = \iint |F(u,v)|^2 \,du\,dv$$

This theorem is often encountered in physical arguments involving conservation of energy - energy must be the same in either domain.

A more general version of this theorem is

$$\iint f(x,y)\, g^*(x,y)\, dx\,dy = \iint F(u,v)\, G^*(u,v)\, du\,dv$$

which can be proved from the autocorrelation theorem.

Parseval's Theorem in 2-D

Parseval's theorem is another way of looking at energy in multiple domains. It applies to functions which are periodic and thus $\int_{-\infty}^{\infty} f^2(t)\,dt$ does not converge.

Suppose $f(x,y)$ is periodic in both x and y with unit period. Then its transform is expressible as a series of coefficients at integral locations in the u,v plane:

$$F(u,v) = \sum\sum a_{mn} {}^2\delta(u-m, v-n)$$

An example of this would be a <u>Fourier series</u>.

<u>Parseval's theorem states</u>

if $f(x+1, y+1) = f(x,y)$ for all x, y (ie, function periodic)

$$\int_{-\frac{1}{2}}^{1/2}\int_{-\frac{1}{2}}^{1/2} |f(x,y)|^2\,dx\,dy = \sum\sum a^2_{mn}$$

or the energy in one period is the sum of the transform coefficients.

Derivative Theorem

$$f(x,y) \stackrel{2}{\supset} F(u,v) \Rightarrow \frac{\partial}{\partial x} f(x,y) \stackrel{2}{\supset} i2\pi u\, F(u,v)$$

In other words, each component of the transform will be scaled by its spatial frequency and shifted $90°$ in phase. So for a single component

$$A \sin 2\pi ux \sin 2\pi yv$$

its derivative is

$$2\pi u \; A \; \sin\left(2\pi u x + \frac{\pi}{2}\right) \sin 2\pi v y$$

Higher frequency components receive higher weight, as we would expect.

Many associated results follow:

$$\frac{\partial}{\partial y} f(x,y) \stackrel{2}{\supset} i\, 2\pi v \, F(u,v)$$

$$\frac{\partial^2}{\partial x^2} f(x,y) \stackrel{2}{\supset} -4\pi^2 u^2 \, F(u,v)$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f(x,y) \stackrel{2}{\supset} -4\pi^2 (u^2+v^2) \, F(u,v)$$

Difference Theorem

For discrete data the equivalent of the derivative is the first difference:

$f(x,y) \stackrel{2}{\supset} F(u,v)$:

$${}^{a}\Delta_x f(x,y) = f(x+\tfrac{1}{2}a, y) - f(x-\tfrac{1}{2}a, y) \stackrel{2}{\supset} 2i \sin \pi a u \, F(u,v)$$

2nd difference:

$${}^{a}\Delta_{xx} f(x,y) = f(x+a,y) - 2f(x,y) + f(x-a,y) \stackrel{2}{\supset} -4 \sin^2 \pi a u \, F(u,v)$$

and so forth.

Definite Integral Theorem

$$f(x,y) \supset F(u,v) \Rightarrow \iint_{-\infty}^{\infty} f(x,y)\,dx\,dy = F(0,0)$$

First Moment Theorem

$$f(x,y) \supset F(u,v) \Rightarrow \iint_{-\infty}^{\infty} x\, f(x,y)\,dx\,dy = \frac{1}{-2\pi i} \frac{\partial}{\partial u} F(0,0)$$

derivative in u of F evaluated at $0,0$

This is proven using the inverse of the derivative theorem:

$$-i2\pi x\, f(x,y) \supset \frac{\partial}{\partial u} F(u,v)$$

negative sign from change of sign in inverse transform

Second Moment Theorems

$$f(x,y) \supset F(u,v) \Rightarrow \iint_{-\infty}^{\infty} x^2 f(x,y)\,dx\,dy = -\frac{\partial^2}{\partial u^2} F(0,0) \cdot \frac{1}{4\pi^2}$$

and a host of others (see text p. 165)

Equivalent area

We can define the equivalent area of a function in two dimensions such that

eq. area $\times$ height of function at origin = volume of function

or

$$A_e \cdot f(0,0) = \iint_{-\infty}^{\infty} f(x,y)\,dx\,dy$$

Similarly the transform $F(u,v)$ has an equivalent area A_F.

The theorem simply states $A_f = \frac{1}{A_F}$

Separable Product Theorem

If a function is separable then its tranform is determined by 2 1-D transforms.

$$f(x) \supset F(u) \text{ and } g(y) \supset G(v) \Rightarrow$$

$$f(x)\, g(y) \overset{2}{\supset} F(u)\, G(v)$$

A special case: $g(y) = 1$, then

$$f(x) \overset{2}{\supset} F(u)\, \delta(v)$$

The Hartley Transform

Related to the Fourier transform is the Hartley transform

$$H(u,v) = \iint_{-\infty}^{\infty} f(x,y)\, \mathrm{cas}[2\pi(ux+vy)]\, dx\, dy$$

$$f(x,y) = \iint_{-\infty}^{\infty} H(u,v)\, \mathrm{cas}[2\pi(ux+vy)]\, du\, dv$$

where $\mathrm{cas}\,\theta = \cos\theta + \sin\theta$

The advantage of the Hartley transform is that the transform of a real image is also real. This can be an advatage for computer implementations.

Hartley Transform Theorems

Affine theorem (incorporates many special cases)

$$f(x,y) \overset{2}{\underset{H}{\supset}} H(u,v)$$

$$f(ax+by+c,\ dx+ey+f) \overset{2}{\underset{H}{\supset}} |\Delta|^{-1} \left[H(\alpha,\beta)\cos\theta - H(-\alpha,-\beta)\sin\theta\right]$$

where $\Delta = ae - bd$

$$\alpha = (eu - dv)/\Delta$$

$$\beta = (-bu + av)/\Delta$$

$$\theta = 2\pi \Delta^{-1}\left[(ec - bf)u + (af - cd)v\right]$$

Conversion theorem

$$f(x,y) \overset{2}{\supset} F(u,v) = R(u,v) + i\,I(u,v)$$

then $f(x,y) \overset{2}{\underset{H}{\supset}} R(u,v) - I(u,v)$

Discrete Fourier Transform

Given for completeness:

$$F(\sigma,\tau) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) \exp\left[-i\left(\frac{2\pi\sigma x}{M} + \frac{2\pi\tau y}{N}\right)\right]$$

and

$$f(x,y) = \sum_{\sigma=0}^{M-1} \sum_{\tau=0}^{N-1} F(\sigma,\tau) \exp\left[i\left(\frac{2\pi\sigma x}{M} + \frac{2\pi\tau y}{N}\right)\right]$$