The Two-Dimensional Fourier Transform

We now begin to discuss the generalization of the Fourier transform to 2-D. Once again, we will find that the variety of phenomena that can occur in higher dimensions greatly exceeds that for 1-D.

As in one-D, we will analyze data into a set of orthonormal sinusoids, but here the sinusoids can be aligned in any direction. We will use the principle of linearity to understand the relationship between the sine and cosine components and the initial image distribution.

Review and reminder – 1-D F.T.

We'll use the following definition of the Fourier transform in one dimension:

$$F(s) = \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi sx}\, dx$$

This can be considered an analysis operation: the component of $f(x)$ with spatial frequency s is selected by the operation. Calculating this for all values of s results in a breaking down, or analysis, of the function for all spatial frequencies.

The inverse procedure, or synthesis, is

$$f(x) = \int_{-\infty}^{\infty} F(s)\, e^{+i2\pi sx}\, ds$$

where we think of reconstructing $f(x)$ from the components derived previously.

Fourier components in 2 dimensions

Similarly, in 2-D we have:

$$F(u,v) = \iint_{-\infty}^{\infty} f(x,y)\, e^{-i2\pi(ux+vy)}\, dx\, dy$$

where u and v play the same spatial frequency role as s. Viewing this as analysis, we are breaking up $f(x,y)$ into sinusoids of form $e^{-i2\pi(ux+vy)}$ or, equivalently,

$\cos(2\pi(ux+vy))$ and $\sin(2\pi(ux+vy))$

Addition formulas give the equivalent

$\cos 2\pi ux \cos 2\pi vy$, $\sin 2\pi ux \sin 2\pi vy$, ...

Symmetry: Since $\cos(\,)$ is symmetric about 0 and $\sin(\,)$ is antisymmetric, functions which are symmetric about both axes may be represented using only the $\cos 2\pi ux \cos 2\pi vy$ terms. Other symmetry arguments follow.

We have mentioned that u and v play the same role as s in the 1-D transform. Thus u is the number of wave crests per unit length in the x-direction, and v in the y-direction. The wavelength in the x-direction is then u^{-1}; in the y-direction v^{-1}. Symmetrical functions can be built up only of components in these directions, so that <u>for these functions only</u>

$$F(u,v) = \iint_{-\infty}^{\infty} f(x,y) \cos 2\pi ux \cos 2\pi vy\, dx\, dy, \text{ and}$$

$$f(x,y) = \iint_{-\infty}^{\infty} F(u,v) \cos 2\pi ux \cos 2\pi vy\, du\, dv$$

But for non-symmetrical functions we need the additional terms and use

$$F(u,v) = \iint_{-\infty}^{\infty} f(x,y) e^{-i 2\pi (ux+vy)} \, dx\,dy$$

and
$$f(x,y) = \iint_{-\infty}^{\infty} F(u,v) \, e^{i 2\pi (ux+vy)} \, du\,dv$$

Quilted surface: Because a 2-D Fourier component can be thought of as a surface like a quilt with lines of stitching running parallel to the axes, but with all possible spacings and complex coefficients out front, we can visually break up the image $f(x,y)$ into many pictures like the following:

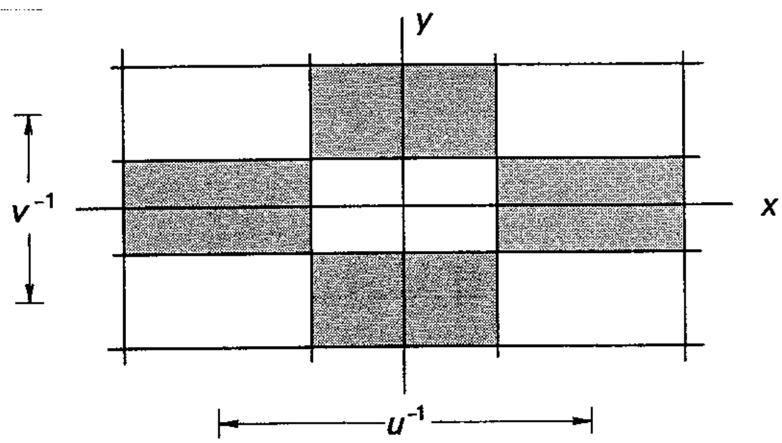


Figure 4-1 The "quilted" surface $\cos 2\pi ux \cos 2\pi vy$. The shaded regions are negative.

Of course we need to consider all possible wavelengths and all possible phase shifts.

3-D representation

The above generalize easily in higher dimensions, but now the surfaces generalize into volumes with sinusoidal spatial distributions. Examples of these include 3-D structures like crystals, which have repeating patterns with possibly different wavelengths in the different directions. If we consider time-varying phenomena 4-D transforms begin to have meaning.

For these cases we might use a compact vector notation: $f(\underline{x}) = f(x,y,z)$, $F(\underline{u}) = F(u,v,w)$, for example. Then

$$F(\underline{u}) = \iiint f(\underline{x})\, e^{-i2\pi(\underline{u}\cdot\underline{x})}\, d\underline{x}$$

$$f(\underline{x}) = \iiint F(\underline{u}) = e^{i2\pi(\underline{u}\cdot\underline{x})}\, d\underline{u}$$

Note the differentials $d\underline{x}$ and $d\underline{u}$ are not vectors, but the area or volume elements $dx\,dy\,dz$ or $du\,dv\,dw$.

Components as Corrugations

It is always helpful to have another viewpoint for abstract quantities, and another way to look at 2-D Fourier components is to view each as a corrugation of the surface with a specific amplitude, wavelength, direction, and phase. Phase may be combined with amplitude if complex coefficients are permitted.

In this case a wavelength q^{-1} is defined according to

$$q = \sqrt{u^2+v^2}$$

and a direction ϕ according to

$$\phi = \tan^{-1}\frac{v}{u}$$

then the component may be viewed as

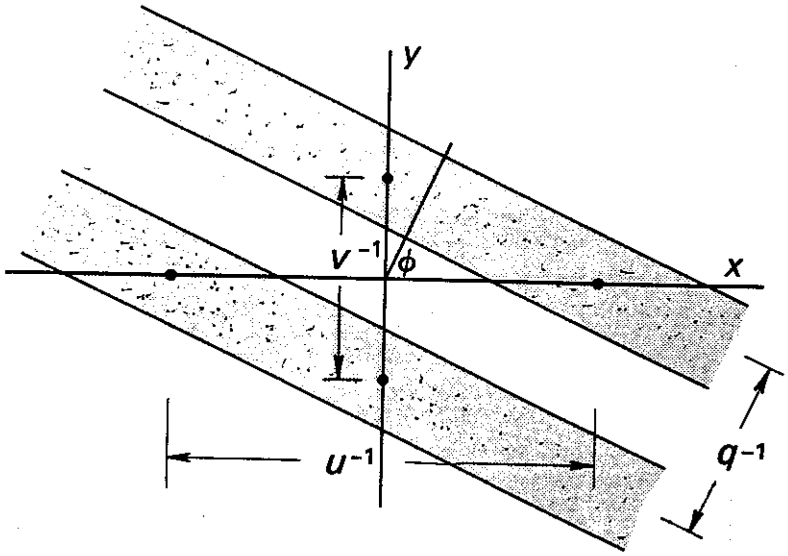


Figure 4-2 A "corrugation" $\cos[2\pi(ux + vy)]$. The shaded zones are negative. The spatial frequency in the x-direction is u (in the y-direction, v); *the* spatial frequency is q.

Now q is the spatial ~~wavelength~~ frequency and ϕ is the orientation with respect to the x-axis. Alternatively,

$$u = q \cos \phi \quad \text{and} \quad v = q \sin \phi$$

The corrogation viewpoint is probably the easiest way to understand the significance of u and v. Consider this picture:

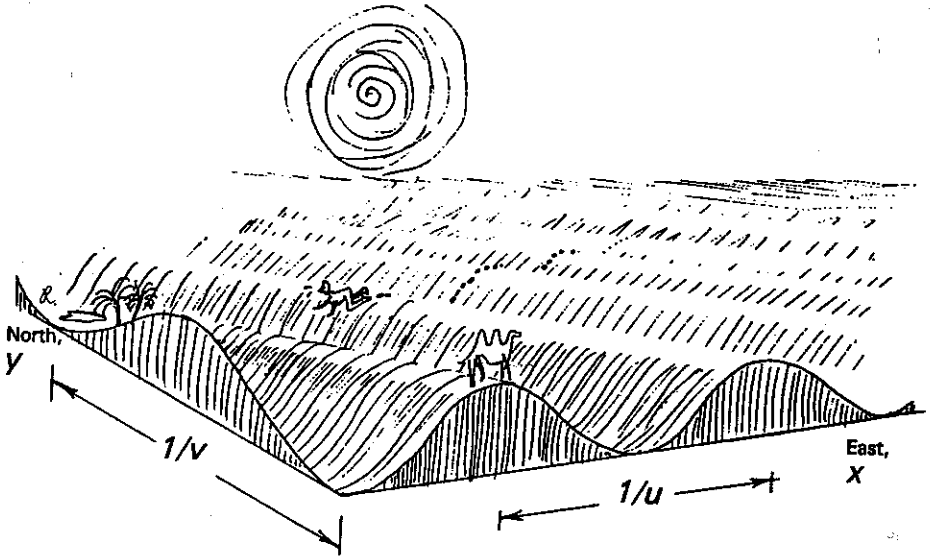
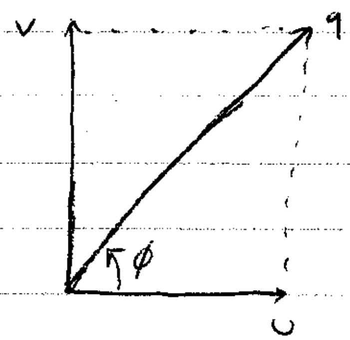


Figure 4-3 The x-component of spatial frequency is u, measured in cycles per unit of x. The spatial period, or crest-to-crest distance traveling east, is u^{-1} (see camel). The spatial frequency q is measured in the direction of hardest going (see man).

This picture is a desert with evenly-spaced dunes at an arbitrary alignment. The camel, traveling due east, experiences a variation in height with wavelength u^{-1}. Going directly north, you would cover dunes with a ~~[illegible]~~ wavelength v^{-1}. But a man crawling in the direction of the dune field sees a wavelength $q^{-1} = (u^2+v^2)^{-1/2}$. The frequency is greatest in this direction.

The numerical interrelation between u, v, and q follow from the vector addition procedure:



2-D Transform pairs

As in 1-D, it is helpful to keep at hand transform pairs which commonly occur. Let's create such a list for 2-D:

Impulse

$$^2\delta(x,y) \supset 1$$

This follows from the sifting property $\iint_{-\infty}^{\infty} {}^2\delta(x,y)\, f(x,y)\,dx\,dy = f(0,0)$

so

$$\iint {}^2\delta(x,y)\, e^{-i2\pi(ux+vy)}\,dx\,dy = 1$$

The inverse transform is then

$$\iint_{-\infty}^{\infty} e^{i2\pi(ux+vy)}\,du\,dv = {}^{2}\delta(x,y)$$

We could "prove" this by invoking the known relationship between the forward and inverse formulas, or we could use a sequence $e^{-\pi\tau^2(x^2+y^2)}$ as $\tau \to 0$. This function has an ~~analy~~ analytical transform and we can evaluate the limit

Pictorially:

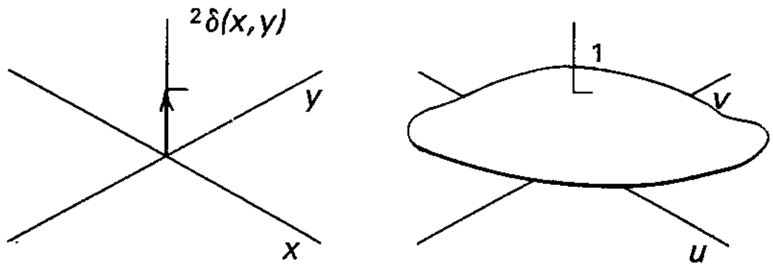


Figure 4-6 The unit two-dimensional impulse and its two-dimensional Fourier transform. The small ticks in this and other illustrations mark unit value along the axes.

Impulse pair. A pair of impulses of $\frac{1}{2}$ strength each, equally spaced from the origin, leads to

$$0.5\,{}^{2}\delta(x+a,y) + 0.5\,{}^{2}\delta(x-a,y) \supset \cos 2\pi a u$$

Inverting the sign of one impulse gives a sine, rather than cosine, function. Viewing this pictorially:

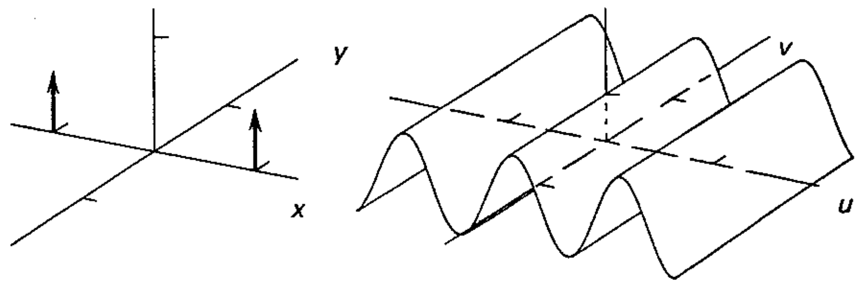


Figure 4-7 Symmetrical impulse pair transforms into a cosinusoidal corrugation. A related example appears in Fig. 4-15.

Here we can easily evaluate the forward transform

$$\iint \left[\tfrac{1}{2}\delta(x+a, y) + \tfrac{1}{2}\delta(x-a, y)\right] e^{-i2\pi(xu+yv)}\, du\, dv$$

but the inverse integral is tough. This is quite common, and we can often save ourselves time by determining which direction is easy and using our knowledge of the forward/inverse transform relationship.

2-D pairs of impulses show up whenever we consider interference phenomena.

Gaussian hump

The 2-D Gaussian $e^{-\pi(x^2+y^2)} = e^{-\pi r^2}$ is its own transform:

$$e^{-\pi r^2} \supset e^{-\pi q^2}$$

If we let r be the radial component in x, y and q the radial frequency in u, v

$$r^2 = x^2+y^2, \quad q^2 = u^2+v^2$$

The coefficient π is included to keep the volume under $e^{-\pi r^2}$ unity. This integral is obtainable in ~~a~~ closed form from a standard table.

Rect (·)

The 2-D rect ${}^2\text{rect}(x,y) = \text{rect}\, x \;\text{rect}\, y$ transforms into the product of sincs:

$$ {}^2\text{rect}(x,y) \supset \text{sinc}\, u \;\text{sinc}\, v $$

The Fourier integral is separable and analytically obtained. We often need to model the radiation pattern of a square or rectangular aperture - this means we often use the square of this transform $\text{sinc}^2 u \,\text{sinc}^2 v$ in antenna problems.

Pillbox or rect r

The transform of the pillbox rect (r) is the ~~j~~ jinc function $\text{jinc}\, q = \frac{J_1(\pi q)}{2q}$ where J_1 is the Bessel function of the first kind of order 1. This is the transform of a circular aperture and shows up very often. We'll discuss the properties of ~~sinc~~ jinc later, for now:

$$ \text{rect}\, r \supset \text{jinc}\, q $$

Rectangular and Circular apertures and their transforms:

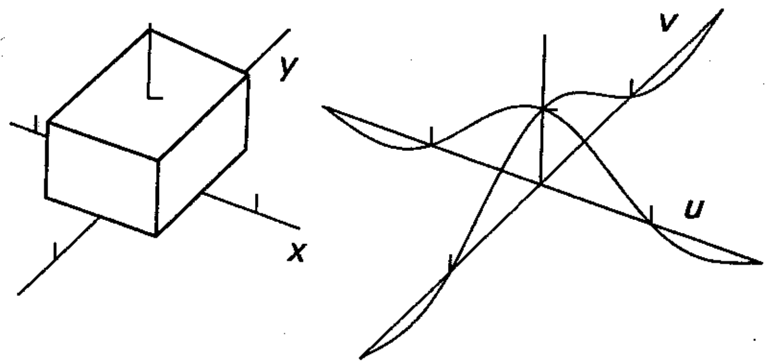


Figure 4-9 Unit two-dimensional rectangle function ${}^2\mathrm{rect}(x, y) = \mathrm{rect}\, x \,\mathrm{rect}\, y$ transforms into a function $\mathrm{sinc}\, u \,\mathrm{sinc}\, v$ that is suggested by its two principal cross-sections.

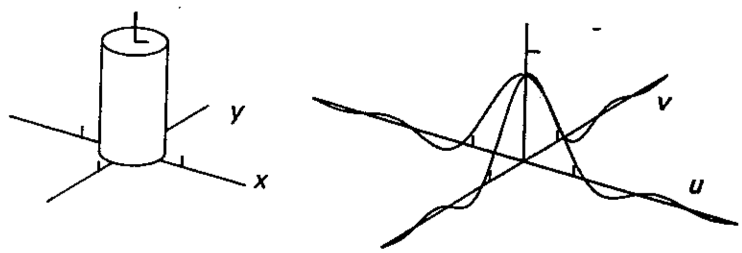


Figure 4-10 Introducing the jinc function, the circular analogue of the sinc function and two-dimensional Fourier transform of the unit pillbox function $\mathrm{rect}\, r$. Heights are exaggerated by a factor 2.

Gaussian ridge

Some function vary in only one dimension, for example, as $e^{-\pi x^2}$. This might be a back projection of a Gaussian along the y-axis. Since the function is constant in the y-direction, not surprisingly a δ-function is involved in its transform:

$$e^{-\pi x^2} \;{}^2\!\supset\; e^{-\pi u^2}\,\delta(v)$$

Here we had to use the ${}^2\!\supset$ symbol to avoid confusion in the dimensionality of the transform.

Its sketch:

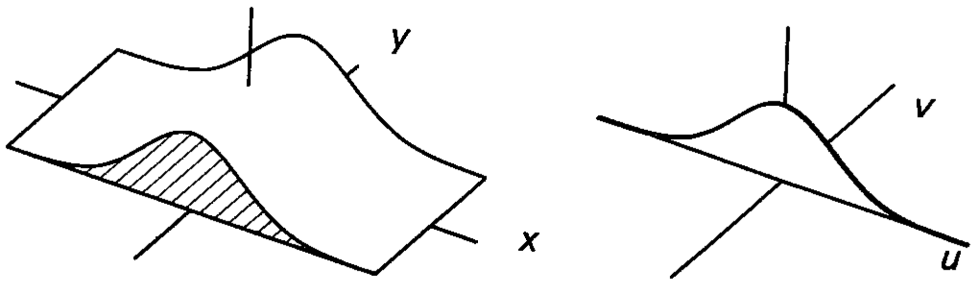


Figure 4-11 The Gaussian cylinder $\exp(-\pi x^2)$ transforms into a Gaussian blade.

Line impulse

Note that if we consider a sequence of the above where we use $\tau^{-1} \exp(-\pi x^2/\tau^2)$ as $\tau \to 0$ we would be defining the transform of $^2\delta(x)$. Thus the following pair is arrived at:

$$^2\delta(x) \;{}^2\!\supset\; {}^2\delta(v)$$

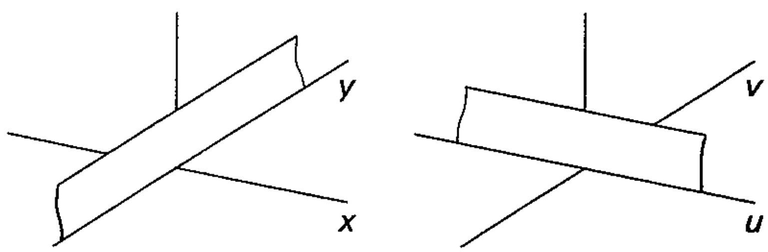


Figure 4-12 A unit line impulse on the y-axis transforms into a unit line impulse on the u-axis.

2-D Signum function

$\mathrm{sgn}(x) = +1$ for positive x, -1 for negative x.

In 2-D it is positive in the first and third quadrants, negative in the second and fourth.

$$^2\mathrm{sgn}(x,y) \;{}^2\!\supset\; \frac{1}{\pi^2 uv}$$

Dipole:

$$\sin 2\theta \;{}^2\!\supset\; {}^2\delta(u,v)\,\sin 2\phi$$

Gaussian with angle dependence

This reciprocal pair appears as an eigenfunction of the ~~Radon~~ Radon transform, which appears in tomographic analysis:

$$e^{-\pi r^2}\sin 2\theta \;{}^2\!\supset\; e^{-\pi q^2}\sin 2\phi$$

Shah function:

In one-d we had $\text{Ш}(x) \supset \text{Ш}(s)$, same in 2-D

$${}^2\text{Ш}(x,y) \;{}^2\!\supset\; {}^2\text{Ш}(u,v)$$

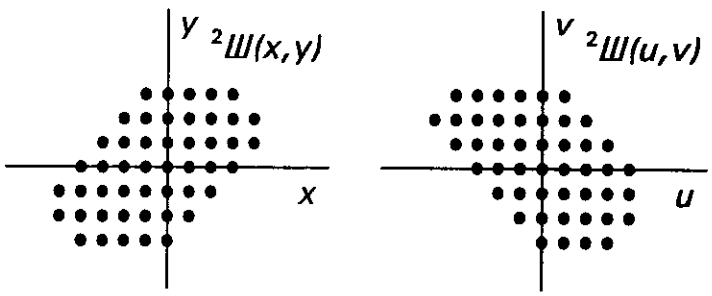


Figure 4-13 The two-dimensional bed-of-nails function $^2\text{III}(x, y)$ is its own transform.

Grid or grille

In 2-D

$${}^2\text{Ш}(x) = \sum_{-\infty}^{\infty} {}^2\delta(x-n)$$ represents a series of horizontal lines spaced at unit ~~intervals~~ intervals. We find

$${}^2\text{Ш}(x) \;{}^2\!\supset\; \text{Ш}(u)\,\delta(v)$$

and

$$^2\mathrm{III}(y) \supset {}^2\mathrm{III}(v)\,\delta(u)$$

This notation is complicated, but the picture is something like

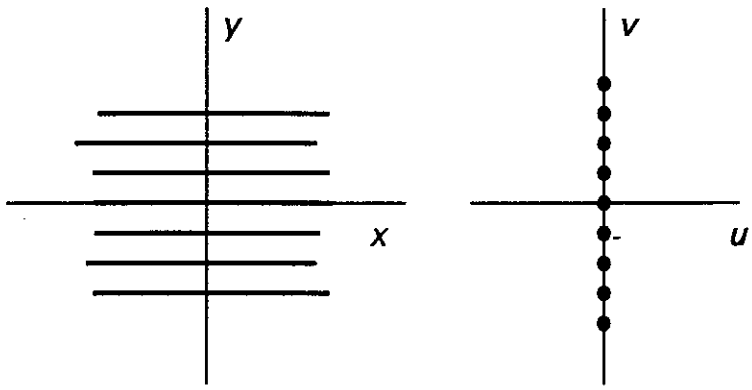


Figure 4-14 The uniform grid III(y) that appears in raster sampling transforms into the vertical impulse row III(v)$\delta(u)$.

This may be used, say, to consider raster-scan data such as a TV signal.

Notes on transform of sgn(x,y)

1-D case: $\mathrm{sgn}(x) = \begin{cases} +1 & x>0 \\ -1 & x<0 \end{cases}$

$$\text{Transform} = \int_{-\infty}^{\infty} \mathrm{sgn}(x)\, e^{-i2\pi sx}\, dx$$

$$= \int_{-\infty}^{0} -e^{-i2\pi sx}\, dx + \int_{0}^{\infty} e^{-i2\pi sx}\, dx$$ ← not integrable because not absolutely convergent

Trick: note if $f(x) = \mathrm{sgn}(x)$, then define $g(x) = e^{-\tau|x|} f(x)$

so $\lim_{\tau\to 0} g(x) = f(x)$

Calculate transform of $g(x)$ and take limit:

$$\text{Transform of } g(x) = \int_{-\infty}^{\infty} e^{-\tau|x|}\, \mathrm{sgn}(x)\, e^{-i2\pi sx}\, dx$$

$$= \int_{-\infty}^{0} -e^{(\tau - i2\pi s)x}\, dx + \int_{0}^{\infty} e^{-(\tau + i2\pi s)x}\, dx$$

$$= \left.\frac{-e^{(\tau - i2\pi s)x}}{\tau - i2\pi s}\right|_{-\infty}^{0} + \left.\frac{e^{-(\tau + i2\pi s)x}}{-(\tau + i2\pi s)}\right|_{0}^{\infty}$$

$$= \frac{-1}{\tau - i2\pi s} + \frac{1}{\tau + i2\pi s}$$

at $\tau = 0$:

$$= \frac{1}{i2\pi s} + \frac{1}{i2\pi s} = \frac{1}{i\pi s} \Rightarrow \mathrm{sgn}(x) \supset \frac{1}{i\pi s}$$