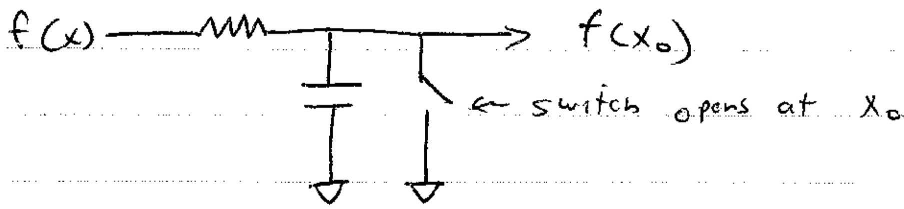
Sifting property -

Sample + hold analog:



$$f(x_0) = \frac{1}{\tau}\int_{x_0-\frac{\tau}{2}}^{x_0+\frac{\tau}{2}} f(x)\,dx$$

Consider δ- function sifting property.

$$f(x_0) = \int \delta(x-x_0)\, f(x)\, dx$$

$$= \int \frac{1}{\tau}\,\mathrm{rect}\left(\frac{x-x_0}{\tau}\right) f(x)\, dx$$

$$= \frac{1}{\tau}\int_{x_0-\frac{\tau}{2}}^{x_0+\tau/2} f(x)\, dx \quad \checkmark$$

Strength of 1-D $\delta()$:

We know $\int_{-\infty}^{\infty} \delta(x)\,dx = 1$

Using 3-step rule:

1. Let $\delta(x) = \frac{1}{\tau}\,\mathrm{rect}\left(\frac{x}{\tau}\right)$

2. $$\int_{-\infty}^{\infty} \frac{1}{\tau}\,\mathrm{rect}\left(\frac{x}{\tau}\right) dx$$

$$= \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} dx = \frac{1}{\tau}\left[\tau/2 - (-\tau/2)\right] = 1$$

3. limit as $\tau \to 0$ integral still unity

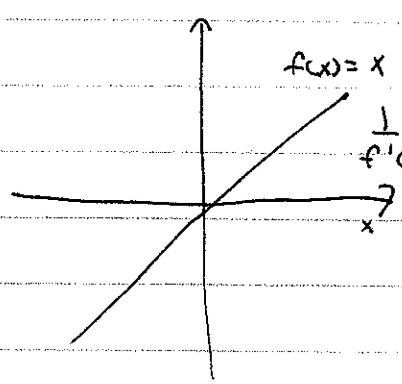
Now consider $\int_{-\infty}^{\infty} \delta(2x)\,dx$

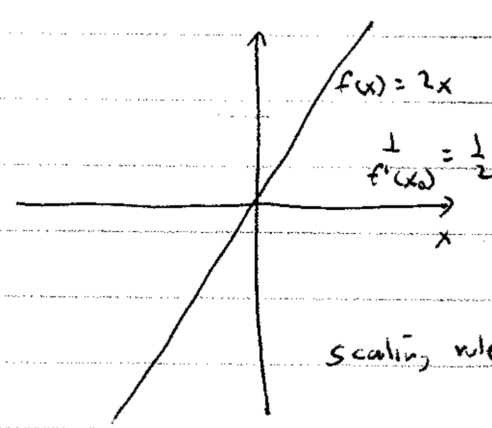
1. $\delta(2x) = \frac{1}{\tau}\,\mathrm{rect}\left(\frac{2x}{\tau}\right)$

2. $\int_{-\infty}^{\infty} \frac{1}{\tau}\,\mathrm{rect}\left(\frac{2x}{\tau}\right) dx$

$$= \frac{1}{\tau}\int_{-\tau/4}^{\tau/4} dx = \frac{1}{2}$$

3. limit as $\tau \to 0$ still $\frac{1}{2}$





strength $= \frac{1}{f'(x_0)}$

scaling rule: $\delta(ax) = \frac{1}{|a|}\delta(x)$

Scaling rule:

$$\delta\left(\frac{x}{a}\right) \approx \frac{1}{\tau}\,\mathrm{rect}\left(\frac{x}{a\tau}\right) \quad \begin{cases} 1 & \left|\frac{x}{a\tau}\right| < \frac{1}{2} \\ 0 & \left|\frac{x}{a\tau}\right| > \frac{1}{2} \end{cases}$$

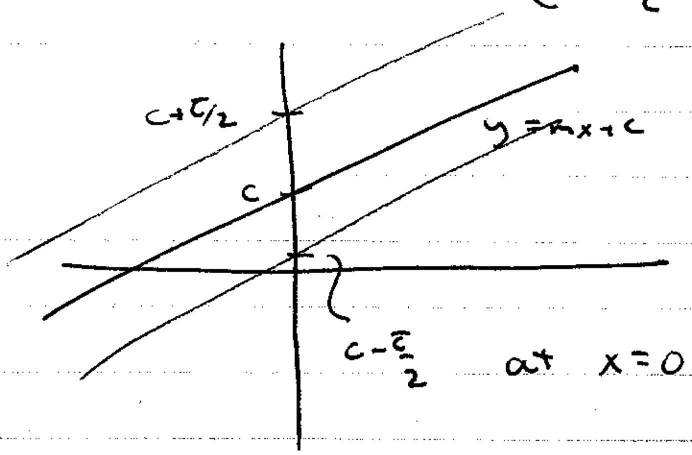
Let $\tau' = a\tau$

$$= \frac{|a|}{\tau'}\,\mathrm{rect}\left(\frac{x}{\tau'}\right)$$

$$\approx |a|\,\delta(x)$$

Consider line impulse $\delta(y - mx - c)$

$$\delta(\,) \rightarrow \frac{1}{\tau}\,\text{rect}\left(\frac{y-mx-c}{\tau}\right)$$



$$\text{rect}\left(\frac{y-mx-c}{\tau}\right) = 1 \quad \left|\frac{y-mx-c}{\tau}\right| < \frac{1}{2}$$

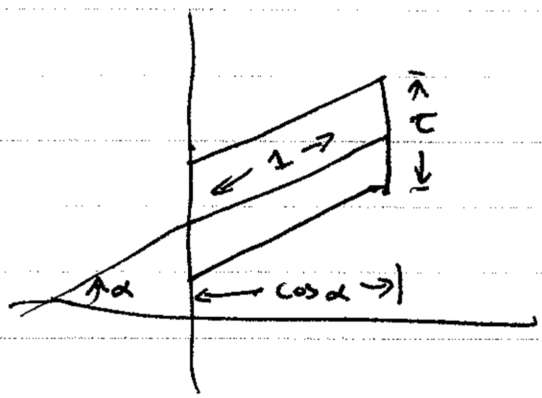
at $x = 0$:

$$\left|\frac{y-c}{\tau}\right| < \frac{1}{2}$$

$$y - c < \frac{\tau}{2}$$

$$y < c + \frac{\tau}{2}$$

so strength / unit length is volume



Area $= \tau \cos\alpha$

volume $= \tau^{-1}\,\tau\cos\alpha = \cos\alpha$

↑ strength / unit length

$\tan \cancel{\alpha} = m$

$1 + \tan^2\alpha = \sec^2\alpha$

$\Rightarrow \cos\alpha = \dfrac{1}{\sqrt{1+m^2}}$

Now, let line be $ly + mx - c$ or $\delta(ly + mx - c)$

$$\text{rect}\left(\frac{ly+mx-c}{\tau}\right) = 1 \text{ if } |ly + mx - c| < \frac{\tau}{2}$$

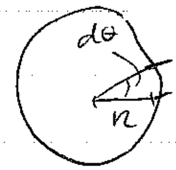
Again at $x = 0$, $ly < c + \tau/2$
$ly > c - \tau/2$

$\Rightarrow$ vertical extent ~~height~~ is $\dfrac{\tau}{l}$

$\Rightarrow$ Area $= \dfrac{\tau}{l}\cdot\cos\alpha$, volume is $\dfrac{\cos\alpha}{l} = \dfrac{1}{l\sqrt{1+\frac{m^2}{l^2}}} = \dfrac{1}{\sqrt{l^2+m^2}} = 1$

$$f(x,y) = \frac{1}{\sqrt{x^2+y^2}} \qquad Q = \int_S S(s)\, f(x,y)\, ds$$

$$\int_0^{2\pi}\int_0^{\infty} \delta(r-R)\, \frac{1}{r}\, r\, dr\, d\theta$$



$$ds = R\, d\theta$$

$$= 2\pi \int_0^{\infty} \delta(r-R)\, dr$$

$$Q = \int_0^{2\pi} 1 \cdot \frac{1}{R} \cdot R\, d\theta$$

$$= 2\pi \int_0^{\infty} \frac{1}{\tau}\, \mathrm{rect}\left(\frac{r-R}{\tau}\right) dr$$

$$\boxed{= 2\pi}\,.$$

$$= \frac{2\pi}{\tau} \int_{R-\frac{\tau}{2}}^{R+\tau/2} dr = \boxed{2\pi}$$

Now, if $\delta(r^2 - R^2)$: still at $r = R$

$$\int_0^{2\pi}\int_0^{\infty} \delta(r^2 - R^2)\, \frac{1}{r}\, r\, dr\, d\theta$$

$$Q = \int_0^{2\pi} \frac{1}{2R} \cdot \frac{1}{R} \cdot R\, d\theta$$

$$= \frac{2\pi}{2R} = \boxed{\frac{\pi}{R}}$$

$$= 2\pi \int_0^{\infty} \delta(r^2 - R^2)\, dr$$

$$= 2\pi \int_0^{\infty} \frac{1}{\tau}\, \mathrm{rect}\left(\frac{r^2 - R^2}{\tau}\right) dr$$

$$= \frac{2\pi}{\tau} \int_{R-\frac{\tau}{4R}}^{R+\tau/4R} dr$$

$$= \frac{2\pi}{\tau} \cdot \frac{\tau}{2R} = \boxed{\frac{\pi}{R}}$$

$\delta(r^2 - R^2)$:

$$\delta(r^2 - R^2) \to \frac{1}{\tau}\,\mathrm{rect}\left(\frac{r^2 - R^2}{\tau}\right)$$

$$\mathrm{rect}\,\frac{r^2 - R^2}{\tau} = 1 \quad \text{if} \quad \left|\frac{r^2 - R^2}{\tau}\right| < \frac{1}{2}$$

$$\text{or} \quad r^2 - R^2 < \frac{\tau}{2}$$

$$r^2 < R^2 \pm \frac{\tau}{2}$$

$$r < \sqrt{R^2 + \frac{\tau}{2}} \approx R\left(1 + \frac{\tau}{2R^2}\cdot\frac{1}{2}\right)$$

$$= R + \frac{\tau}{4R}$$

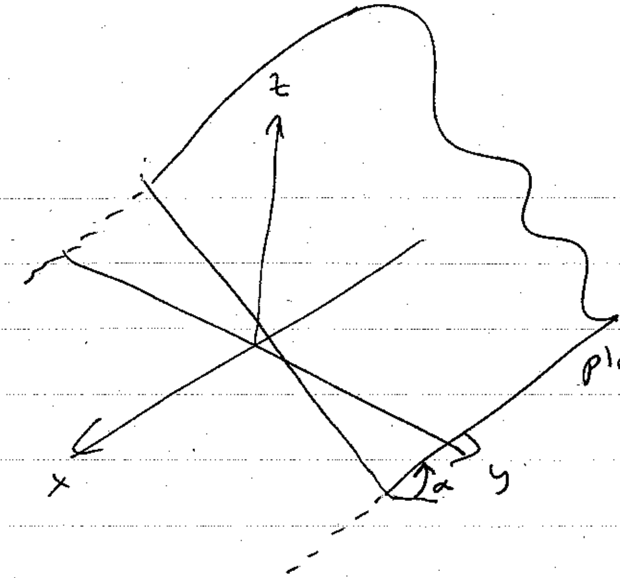
$$\iint \delta(r^2 - R^2)\, r\,dr\,d\theta =$$

$$2\pi \int_{R - \frac{\tau}{4R}}^{R + \tau/4R} \frac{1}{\tau} r\,dr = \frac{2\pi}{\tau}\left[\frac{(R + \tau/4R)^2}{2} - \frac{(R - \tau/4R)^2}{2}\right]$$

$$= \frac{2\pi}{\tau}\left[\frac{\cancel{R^2} + \cancel{\tau^2/16R^2} + \cancel{2}\cdot \cancel{R}\cdot\tau/4\cancel{R} - \cancel{R^2} - \cancel{\tau^2/16R^2} + \cancel{2}\cdot\cancel{R}\cdot\tau/4\cancel{R}}{\cancel{2}}\right]$$

$$= \frac{2\pi}{\tau}\left[\frac{\tau}{2}\right] = \underline{\underline{\pi}}$$

Strength / unit length $= \dfrac{\pi}{2\pi R} = \dfrac{1}{2R}$



plane $z = ly + mx + c$

$\tan\alpha = \text{grad } z$

$$= \left(\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right)^{1/2}$$

$$= \sqrt{l^2 + m^2}$$

$$= 1$$

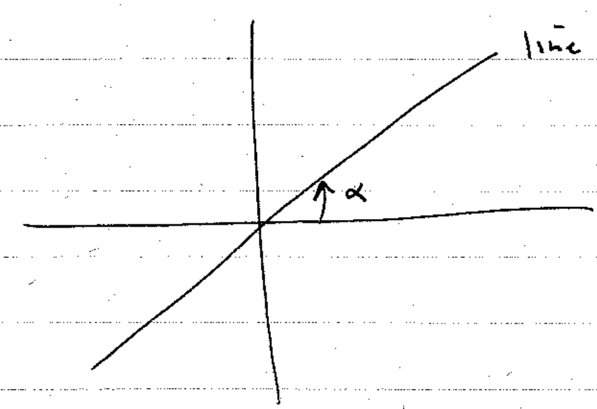
Original case: $z = y - mx - c$

$\text{grad } z = \sqrt{1 + m^2}$

$\text{strength} = \dfrac{1}{\text{grad } z} = \dfrac{1}{\sqrt{1+m^2}}$

$\left\{ \begin{array}{l} m = \tan\alpha \\ 1 + \tan^2\alpha = \sec^2\alpha = \dfrac{1}{\cos^2\alpha} \\ \text{grad } z = \dfrac{1}{\cos\alpha} \\ \text{strength} = \cos\alpha \end{array} \right.$

$\delta(ax) \rightarrow \dfrac{1}{|a|}\delta(x)$



line $y = ax$

$y = \tan\alpha \cdot x$

~~$\delta(ax) \Rightarrow |a|\,\delta(x)$~~

Consider in 1-D $\delta(x)$, $\delta(2x)$

in general $\delta(f(x))$

$\text{strength} = \dfrac{1}{f'(x_0)} = \dfrac{1}{\tan\alpha}$