Regular Impulse Patterns

We have already examined one regular pattern of impulses, ${}^2\text{III}(x,y)$. Several others we commonly use are:

Impulsive Grille:

Sometimes it is convenient to use a set of equally-spaced line impulses. For example, a set of vertical lines might be described by

$$\sum_{n=-\infty}^{\infty} \delta(x-n),$$

which we can see is simply $\text{III}(x)$ interpreted as a 2-D function. Using our 3 rules to interpret $\text{III}(x)$,

$$\text{III}(x) = \sum_{n=-\infty}^{\infty} \tau^{-1}\,\text{rect}\left(\frac{x-n}{\tau}\right) \quad \text{as } \tau \to 0.$$

Row of point impulses:

Different from the two-dimensional shah function, a single row of 2-D δ-functions would be described

$$\sum_{n} {}^2\delta(x-n,y)$$

where in this case the row is located on the x-axis. An alternative expression might be the impulsive grill multipled by a delta

$$\sum_{n=-\infty}^{\infty} {}^2\delta(x-n,y) = \text{III}(x)\,\delta(y)$$

This function can be evaluated by two applications of the 3-step rule, one in x and one in y.

This function shows up in certain sampling applications, and helps in determining functions with periodicity in one direction (not dimension), as:

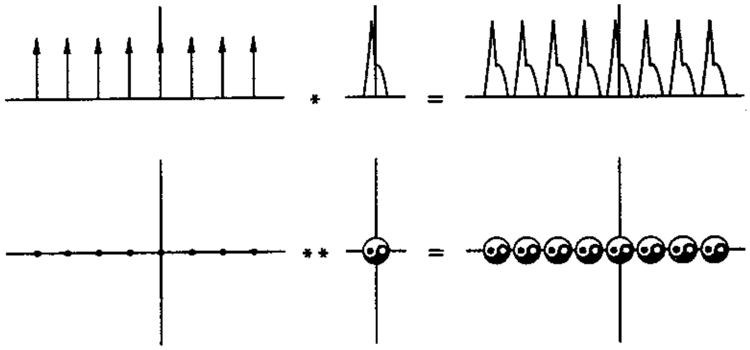
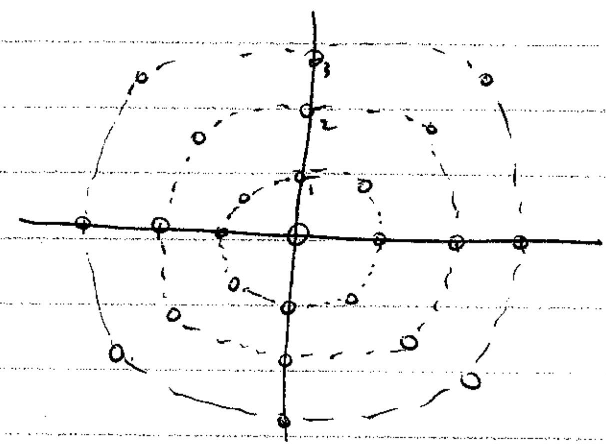


Figure 3-9 The row of impulses and its role in representing regular replication along a straight line, in one dimension (above) and in two dimensions (below).

Although we haven't introduced 2-D convolution yet, the meaning of the above figure is rather plain.

Other patterns

Many other patterns are possible. What is the representation in δ-functions for the following polar coordinate sampling?



Rectangle Function of a Function $f(x)$

So far we have used the notation $\text{rect}(f(x))$ for very simple functions whose meaning is clear. At most a little shifting and scaling was needed - for a function

$$f(t) = h \ \text{rect}\left(k(t-T)\right)$$

we recognize a function of height h, compressed by k in the t-direction, and shifted to the right by T.

For more complex situations, such as rect (sin t) or rect (t^2-1), we may not be able to visualize the results easily, and we have to use the definition to understand:

$$\text{rect}(\sin t) = \begin{cases} 1 & |\sin t| < \frac{1}{2} \\ 0 & |\sin t| > \frac{1}{2} \end{cases}$$

or

$$\text{rect}(t^2-1) = \begin{cases} 1 & |t^2-1| < \frac{1}{2} \\ 0 & |t^2-1| > \frac{1}{2} \end{cases}$$

If we were to plot these rect functions along with their original functions:

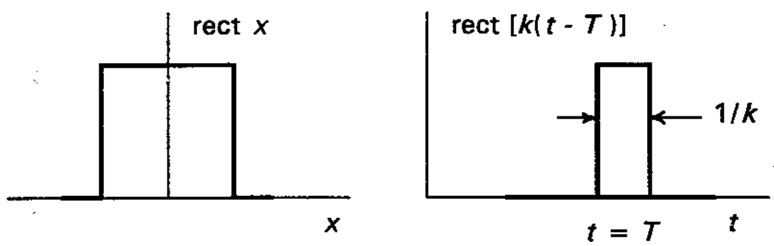


Figure 3-10 The unit rectangle function rect x (left) and a rectangle function of a simple expression (right).

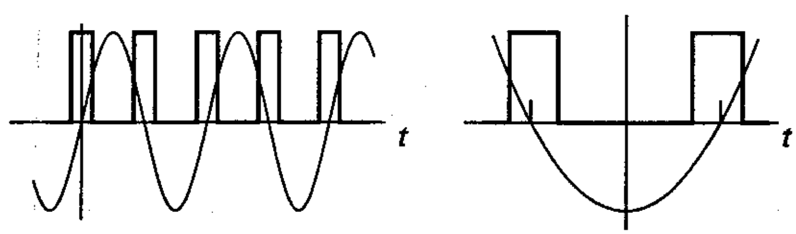


Figure 3-11 Illustrating rectangle functions of a sine wave and of a parabolic waveform.

The rect functions pick intervals about the zero crossings of the original functions. Note that in the case of rect (t^2-1) the rectangles are not centered at the zeroes? Why not?

For a more complicated function, we might have the following situation:

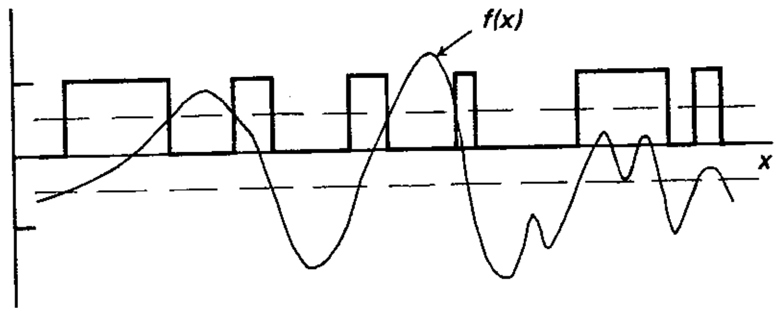


Figure 3-12 Illustrating the rectangle function of an arbitary function.

Now, note that not all zero crossings generate a unique rect $\neq 0$, and much fine structure is lost if we were to try to represent f(x) simply by its rect(f(x)).

Extension to 2-D

The 2-D extension of rect(f(x,y)) follows again from the formal definition:

$$\text{rect}\,(f(x,y)) = \begin{cases} 1 & |f(x,y)| < \frac{1}{2} \\ 0 & |f(x,y)| > \frac{1}{2} \end{cases}$$

The same cautions apply to this case as in the case of the arbitrary 1-D function, except here we have a band around null contours in a 2-D function.

Examination of the arbitrary 1-D function above shows that the "width" of the rect impulses are inversely proportional to the slope of the function f'(x) near the zero crossing. (Why is this?) So for the 2-D case the "width" of the band resulting

resulting from the rect about a zero contour is also inversely proportional to the gradient taken at right angles to the contour. In other words, the steeper the slope, the narrower the rect $(f(x,y))$ band. See the following illustration:

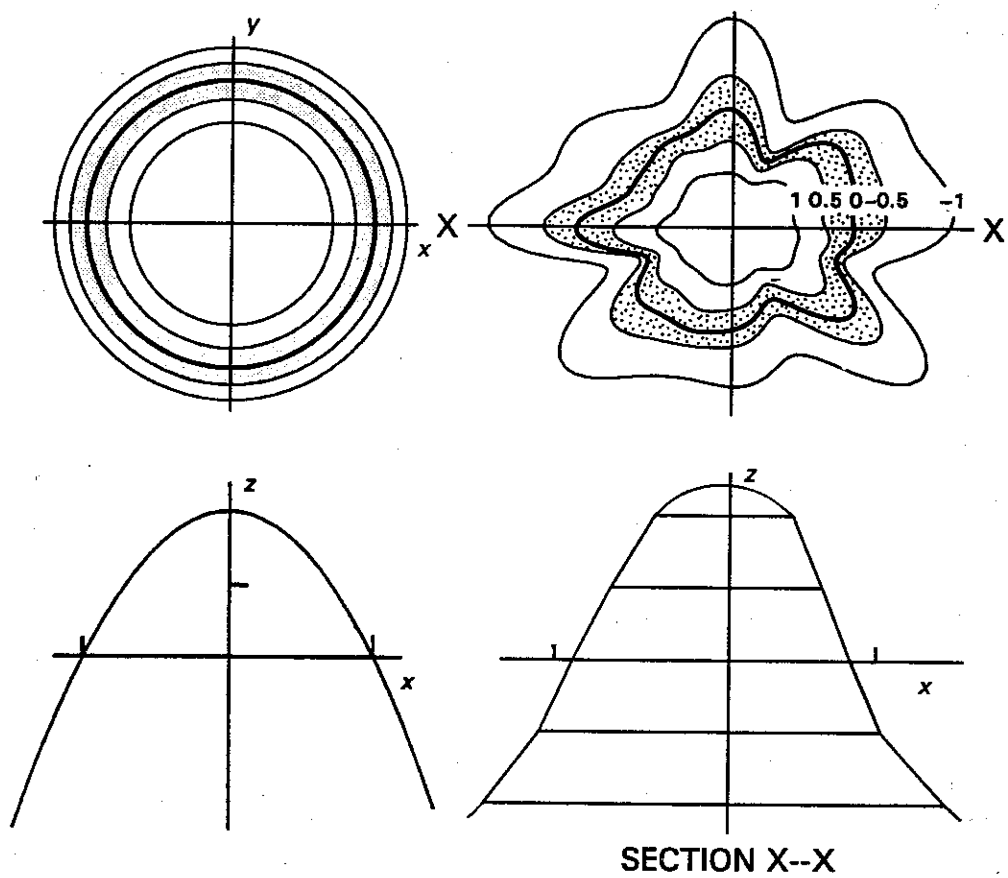


Figure 3-13 Functions $f(x, y)$ (above) with the zone between the contours at ± 0.5 shown shaded. Cross sections (below) show how the width of the shaded zone where it cuts the x-axis is connected with the steepness of the section. In each case rect$[f(x, y)]$ is unity in the shaded zone and zero elsewhere.

The General Rule for Line Deltas

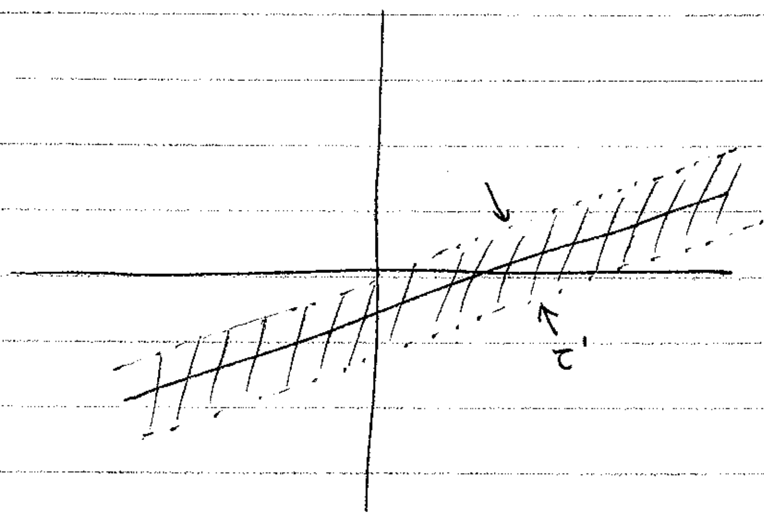
We found earlier that by parameterizing a line by its slope leads to the result that the line delta strength is a function of slope, a consequence of the construction we used to evaluate the strength, or, equivalently, the construction of the physical quantity. In some cases we want to examine a line delta ~~want~~ we can rotate so that we don't have to worry as much about choosing the proper coordinates, or construction.

Represent the line delta as $\delta(lx+my+c)$, a line defined by $lx+my+c=0$. Also assume $l^2+m^2=1$, that is l and m are the direction cosines of the line, the cosines of the angle between the line and the x and y axes, respectively.

We can define a rotated coordinate system (x', y') where the y-axis is parallel to the line, we will see that this corresponds to $x' = lx+my$.

The strength of the line-δ in this coordinate system is then

$$\int_{-\infty}^{\infty} \delta(lx+my+c)\,dx' = \int_{-\infty}^{\infty} \delta(x'+c)\,dx' = 1$$



Here the line delta is defined with the τ' direction at right angles to the line, rather than in the y-direction as before, so the strength will be independent of the angle. So choosing the proper coordinate system is critical in determining the strength of the line-δ.

Ring impulse

Another fundamental function in 2-D is the ring impulse, or ring delta function. This function is zero everywhere except along a circle, a line of constant radius. We can describe the 2-D function by

$$f(x,y) = \delta(r - R)$$

which is a circle of radius R. Again the location is straightforward, so now we evaluate the strength.

Since this function is of finite extent, we can determine either the linear density as before or the total "mass".

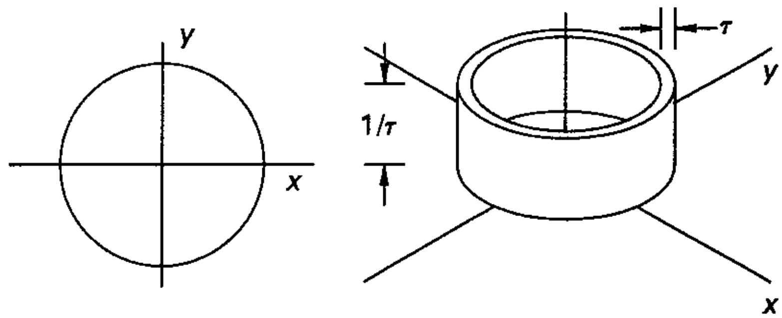


Figure 3-16 The ring impulse on the (x, y)-plane (left) and the circular wall for discussing the ring impulse (right).

Viewing the ring-δ as a mass on a plane, we might examine the relation between expressions of strengths in terms of units:

Representation	Units (mass example)
$f(x,y)$ - Area density	kg m^{-2}
$w(s)$ - Linear density	kg m^{-1}
~~Mass~~ $\iint f(x,y)\,dx\,dy$ - mass	kg
$\int w(s)\,ds$ - mass	kg

Note that the same mass is determined whether we integrate over the area or along the ring-δ. Arc length s is measured along the curve.

To evaluate the strength, remember the rules:

1. $\delta(\cdot) \to \tau^{-1} \operatorname{rect}\left(\frac{\cdot}{\tau}\right)$
2. Evaluate expression
3. Take limit $\tau \to 0$

So, for $\delta(r-R)$:

$$\delta(r-R) \to \tau^{-1} \operatorname{rect}\left(\frac{r-R}{\tau}\right).$$

Referring back to the picture on the previous page, this rect function can be thought of as a circular wall of thickness τ and height τ^{-1}, hence unit cross-section. The inner and outer limits of the wall are at $r-R = \pm \tau/2$ or $r = R \pm \frac{\tau}{2}$.

To calculate the total mass of the ring, we use

$$M = \iint f(x,y)\, dx\, dy$$

$$= \iint \frac{1}{\tau} \operatorname{rect}\left(\frac{r-R}{\tau}\right) dx\, dy$$

Changing to polar coordinates

$$M = \int_0^{2\pi}\int_0^{\infty} \frac{1}{\tau} \operatorname{rect}\left(\frac{r-R}{\tau}\right) r\, dr\, d\theta$$

Integration over θ is easy: $M = \int_0^{\infty} \frac{1}{\tau} \operatorname{rect}\left(\frac{r-R}{\tau}\right) r\, dr \cdot 2\pi$

$$\therefore\ M = 2\pi \int_{R-\frac{\tau}{2}}^{R+\tau/2} \frac{r}{\tau}\, dr = \frac{2\pi}{\tau}\left[\frac{(R+\tau/2)^2}{2} - \frac{(R-\tau/2)^2}{2}\right] = \underline{\underline{2\pi R}}$$

So the total mass of the ring delta is proportional to radius. So for linear density, we divide by line length, which is also $\propto$ to radius, and find that the ring impulse has unit linear density.

Generalization to the impulse function of $f(x,y)$

We can of course use any curve in the x-y plane to define an impulse. The methodology is the same as for the simple ring but the math can be more complicated. Clearly the location of the arbitrary impulse will be at $f(x,y)=0$, and the strengths will follow from the 2-step rule.

Our construction involves building our "wall" with its width τ at right angles to the curve. Recall from our earlier discussions of contour maps that the strength will be inversely to the slope of the surface $f(x,y)$ along the contour, where the maximum slope is found at right angles to the zero contour.

To calculate: we can differentiate in x and y directions and infer the slope in the normal direction $\hat{n}$:

$$\left|\frac{\partial f}{\partial n}\right| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} = |\text{grad } f|$$

The strength is then the reciprocal:

$$\text{strength} = \left|\frac{\partial f}{\partial n}\right|^{-1}_{\text{along } f=0} = \frac{1}{|\text{grad } f|_{\text{along } f=0}}$$

In one-d we found that $\delta(f(x))$ have strengths inversely proportional to the absolute value of the slopes at $f(x)=0$. In 2-D the same is true but we use the gradient instead.

Example. Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which follows from
$f(x,y) = \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}} - 1$.

Calculating the components of the gradient:

$$\frac{\partial f}{\partial x} = \frac{x}{a^2} \frac{1}{f(x,y)+1}$$

$$\frac{\partial f}{\partial y} = \frac{y}{b^2} \frac{1}{f(x,y)+1}$$

This simplifies nicely for the null contour $f(x,y)=0$:

$$\frac{\partial f}{\partial x} = \frac{x}{a^2}$$

$$\frac{\partial f}{\partial y} = \frac{y}{b^2}$$

$|\text{grad } f| = \sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4}}$ and the strength is the reciprocal of this quantity.

A plot of the strength of the impulse might be:

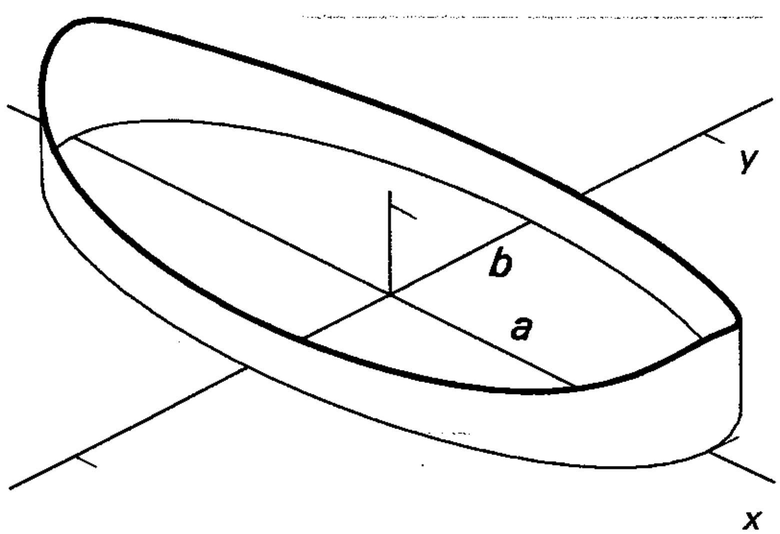


Figure 3-17 The elliptical line impulse $\delta(\sqrt{x^2/a^2 + y^2/b^2} - 1)$ is not of uniform strength.

Intuitively we might understand the variable character of the strength as follows. Consider the slope (gradient) of the function at its axis ends. At the point $x=a, y=0$, the slope is $\frac{1}{a}$, while at $x=0, y=b$ it is $\frac{1}{b}$. The strengths at these points are a and b, ~~respe~~ respectively.

The sifting property

Sampling of a signal is one ~~to~~ of the most common applications of the sifting property, which selects certain values of a function. If we have a unit impulse at location (a,b) is multiplied by a function $f(x,y)$ and integrated over the full plane, the only portion of $f(x,y)$ that contributes is the value at (a,b). Since the strength of the impulse is unity, we have

$$\iint_{-\infty}^{\infty} {}^2\delta(x-a, y-b)\, f(x,y)\, dx\, dy = f(a,b)$$

If the impulse is at the origin,

$$\iint_{-\infty}^{\infty} {}^2\delta(x,y)\, f(x,y)\, dx\, dy = f(0,0)$$

So the operation of multiplying by an impulse and integrating "sifts" out the value of the function at the location of the impulse. To derive this property, we can use the familiar procedure.

$$\iint_{-\infty}^{\infty} \tau^{-2}\, \mathrm{rect}\left(\frac{x-a}{\tau}, \frac{y-b}{\tau}\right) f(x,y)\, dx\, dy$$

$$\approx \int_{a-\frac{\tau}{2}}^{a+\tau/2} \int_{b-\frac{\tau}{2}}^{b+\frac{\tau}{2}} \tau^{-2} f(x,y)\,dx\,dy$$

$$\approx f(a,b)$$

with the approximation getting better and better as $\tau \to 0$, unless the function is discontinuous at a, b. We can visualize this as the volume of a square prism of height $\tau^{-2} f(a,b)$ centered at (a,b)

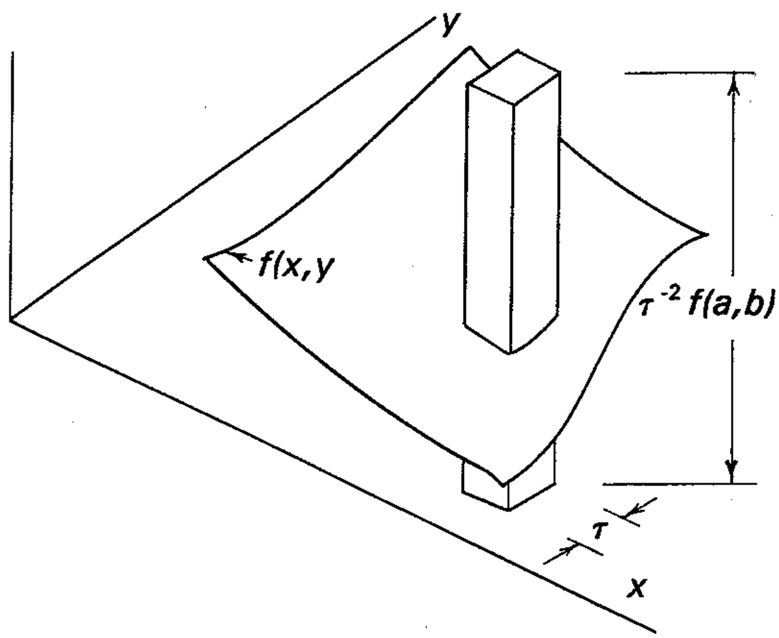


Figure 3-18 The function $\tau^{-2}\ {}^{2}\mathrm{rect}[(x-a)/\tau, (y-b)/\tau] f(x,y)$ has a height about $\tau^{-2} f(a,b)$ and a base area τ^2. Consequently, its volume is approximately $f(a,b)$, and the approximation gets better and better as $\tau \to 0$.

If the function is indeed discontinuous, we could evaluate the integral in pieces and add the results together.

Curvilinear case

The sifting property is a little different when the impulse is a curvilinear line impulse. Here the sifted amount will amount to a line integral along the path weighted by the strength. ~~If the impulse has unit strength,~~ The sifted quanity Q is

$$Q = \int_C S(s)\, f(x,y)\, ds$$

where $S(\cdot)$ is the curvilinear impulse and s is an element of arc; $S(s)$ is the strength along the impulse.

Difference between $\delta(\,)$'s at same location

Lets consider the case of two ring impulses at the same location, one defined by $\delta(r-R)$ and one defined by $\delta(r^2-R^2)$. Both of these are ring impulses defined on a circle of radius R. How do their strengths compare?

We have already looked at $\delta(r-R)$. Its total "mass" was found to be $2\pi R$, leading to a density of $\frac{2\pi R}{2\pi R}$ or unity along the arc.

So, for $\delta(r^2-R^2)$, apply our rules and integrate over the plane to get the total mass.

$$\iint_{-\infty}^{\infty} \delta(r^2-R^2)\,dx\,dy \Rightarrow \iint_{-\infty}^{\infty} \frac{1}{\tau}\,\mathrm{rect}\left(\frac{r^2-R^2}{\tau}\right)dx\,dy$$

$$\mathrm{rect}\,\frac{r^2-R^2}{\tau} = \begin{cases} 1 & \left|\frac{r^2-R^2}{\tau}\right| < \frac{1}{2} \\ 0 & \text{else} \end{cases}$$

so the function is unity for r^2 in between $R^2 \pm \frac{\tau}{2}$ or r in between $\sqrt{R^2 \pm \frac{\tau}{2}}$

For small τ, $\sqrt{R^2 \pm \frac{\tau}{2}} \approx R\left(1 \pm \frac{1}{2}\frac{\tau}{2R^2}\right)$

$$= R \pm \frac{\tau}{4R}$$

$$\text{Mass} = \int_0^{2\pi}\int_{R-\frac{\tau}{4R}}^{R+\tau/4R} \frac{1}{\tau}\, r\,dr\,d\theta$$

$$= \frac{2\pi}{\tau}\left.\frac{r^2}{2}\right|_{R-\frac{\tau}{4R}}^{R+\tau/4R}$$

$$= \frac{2\pi}{2\tau}\left[\left(R^2 + \tau^2/_{16R} + \frac{\tau}{2}\right) - \left(R^2 + \frac{\tau^2}{16R} - \frac{\tau}{2}\right)\right]$$

$$= \underline{\underline{\pi}}$$

which is independent of R. The linear density then is $\frac{\pi}{2\pi R} = \frac{1}{2R}$. Why are the two ring deltas of different strength, even though they are at the same location?

Recall that they are defined by two different underlying functions, which have different slopes and therefore different strengths at the point of the null contour. Evaluate the gradients of $f(x,y) = r - R$ and $g(x,y) = r^2 - R^2$;

In polar coordinates, $\text{grad}_r(f) = \frac{\partial f}{\partial r}$ and $\text{grad}_\theta f = \frac{1}{r}\frac{\partial f}{\partial \theta}$.

The total gradient is

$$\left(\left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2\right)^{1/2}$$

Because we have circular symmetry, the gradient we want is simply $\left|\frac{\partial f}{\partial r}\right|$.

Strength $= |\text{grad}|^{-1}$, so

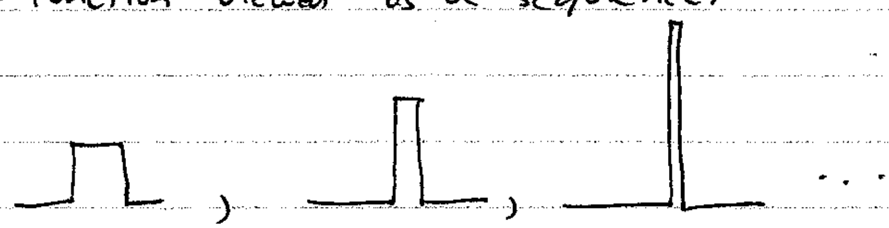
for $f = r - R$, $\frac{\partial f}{\partial r} = 1$ and strength/unit length $= 1$

for $g = r^2 - R^2$, $\frac{\partial g}{\partial r} = 2r$ and strength/unit length $= \frac{1}{2R}$

Remember to consider the shape of the underlying functions, because the slopes, not only the locations, determine the strengths of the impulses.

Intuitive feel for derivatives of impulses

Consider what our $\frac{1}{\tau}\,\text{rect}\left(\frac{x}{\tau}\right)$ representation tells us about a δ-function viewed as a sequence:



Each successive member gets taller and thinner, approaching a limit. Can we construct a similar sequence for the derivative of a δ?

Recall the formal rule for derivatives:

$$f'(x) = \lim_{a \to 0} \frac{f(x) - f(x-a)}{a}$$

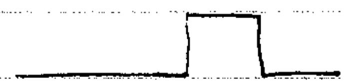
so for our $\delta(\cdot)$:

$$\delta'(x) = \lim_{a \to 0} \frac{\delta(x) - \delta(x-a)}{a}$$

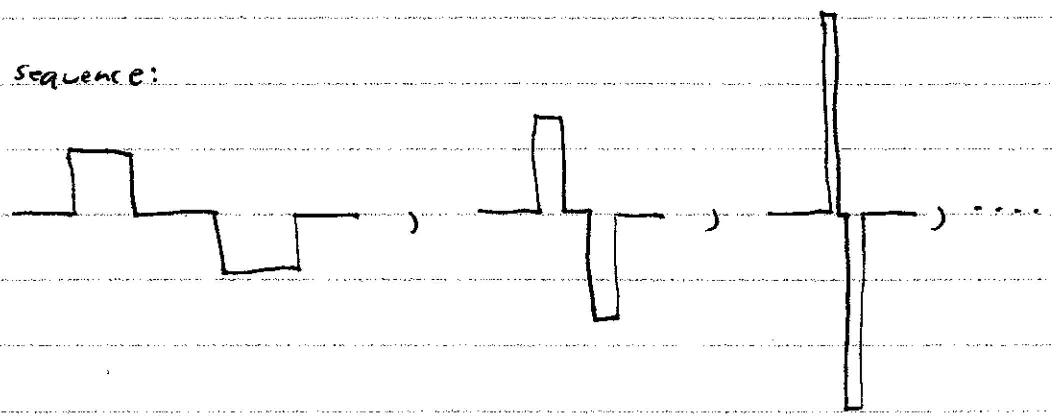
so construct a sequence

$$\lim_{a \to 0} \frac{\frac{1}{\tau} \operatorname{rect} \frac{x}{\tau} - \frac{1}{\tau} \operatorname{rect} \frac{x-a}{\tau}}{a}$$

$\operatorname{rect}\left(\frac{x}{\tau}\right)$ 

$\operatorname{rect}\left(\frac{x-a}{\tau}\right)$ 

Our sequence:



as $\tau \to$ smaller and smaller. So we at least have a way to evaluate expressions requiring $\delta'(x)$. The 2-D analog is quite similar.