The following material starts with Chapter 3 of the text. Read this chapter this week.

Two-dimensional Impulse Functions

In one dimension we often find it useful to have functions which are concentrated at certain points on the number line, for example for sampling functions. The same holds for two dimensions, but now we have the possibility of concentrating signals at either points or lines. Now we will generalize the idea of δ- functions to 2-D.

Two-dimensional Point Impulse

Consider a "point load" in mechanics, or a "point charge" in electrostatics. Each represents an idealization of a physical quantity concentrated at one spot on the two-dimensional plane.

If $p(x,y)$ is the pressure distribution over the (x,y) plane, then the force applied to region R is determined by the integral of the pressure over that area:

$$F = \iint_R p(x,y)\, dx\, dy$$

Now let's suppose that the pressure is concentated at location (a,b) in the plane, and zero everywhere else. We can express a given force F applied at one point in the plane as resulting in a pressure P

$$P = F\, {}^2\delta\,(x-a, y-b)\, dx\, dy$$

We let ${}^2\delta(x,y)$ be the pressure distribution due to a unit force applied at the origin, then $F\,{}^2\delta(x,y)$ is a force F at the origin.

We know the integral of pressure is force, so

$$\iint_{-\infty}^{\infty} F\,{}^2\delta(x,y)\,dx\,dy = F \quad \Rightarrow \iint_{-\infty}^{\infty} {}^2\delta(x,y)\,dx\,dy = 1$$

This is an analog to the 1-D case: $\int_{-\infty}^{\infty} \delta(x)\,dx = 1$

A similar argument holds for point charges:

~~Charge~~ Charge density $\sigma(x,y) = Q\,{}^2\delta(x,y)$

then $\iint_{-\infty}^{\infty} \sigma(x,y)\,dx\,dy = \iint_{-\infty}^{\infty} Q\,{}^2\delta(x,y)\,dx\,dy = Q$

We can easily extend this to three dimensions. Then we would use the volume charge density $\rho(x,y,z)$ and represent the 3-D density as $Q\,{}^3\delta(x,y,z)$.

Point functions Interpreted as sequences

In real life, a "point" load does not result in infinite pressure. The real pressure distribution is spread out over some area. But we can model and calculate effects quite easily with the tools afforded by idealized point forces.

Of course, true infinite point forces would break molecular bonds and destroy the physical objects subject to pressure. But since the integral over area is finite we can still model the process with δ-functions.

Consider the following example of a load applied to a board supported on both ends:

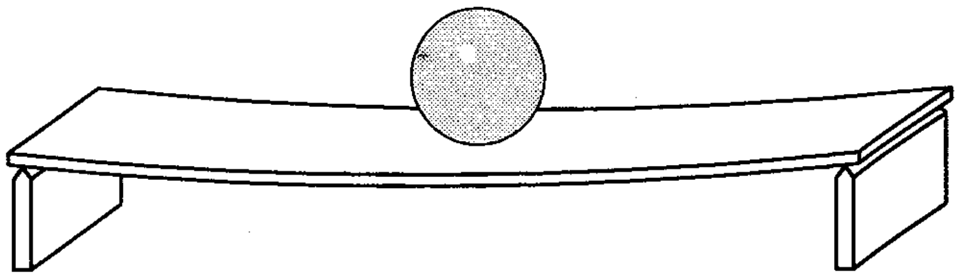
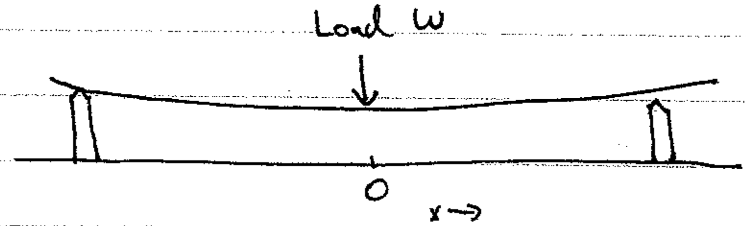


Figure 3-2 A point load W at distance x from the center of a simply supported beam of length L produces a deflection $Wx \times (3L^2 - 4x^2)/48EI$ at a distance x from the left support, where E is the Young's modulus of the material of the beam, I is the moment of inertia of the cross-section, and $0 \leq x \leq L/2$. The exact deformation at the point of contact is not significant in this problem.

We might model the situation as follows:



(*)

Impulse of strength W: $W\delta(x)$

2-D: $W\,{}^2\delta(x,y)$

Now suppose we obtained 3 bricks with the same weight W (when combined) at the original sphere. Depending on how we stack the bricks on the board, we will approach the idealized case closer and closer. How close we need to get depends on properties of the board and sphere, but clearly the deflection can be calculated easily from the mathematical model (*) above.

Let's look at the ways the bricks might be stacked as a sequence:

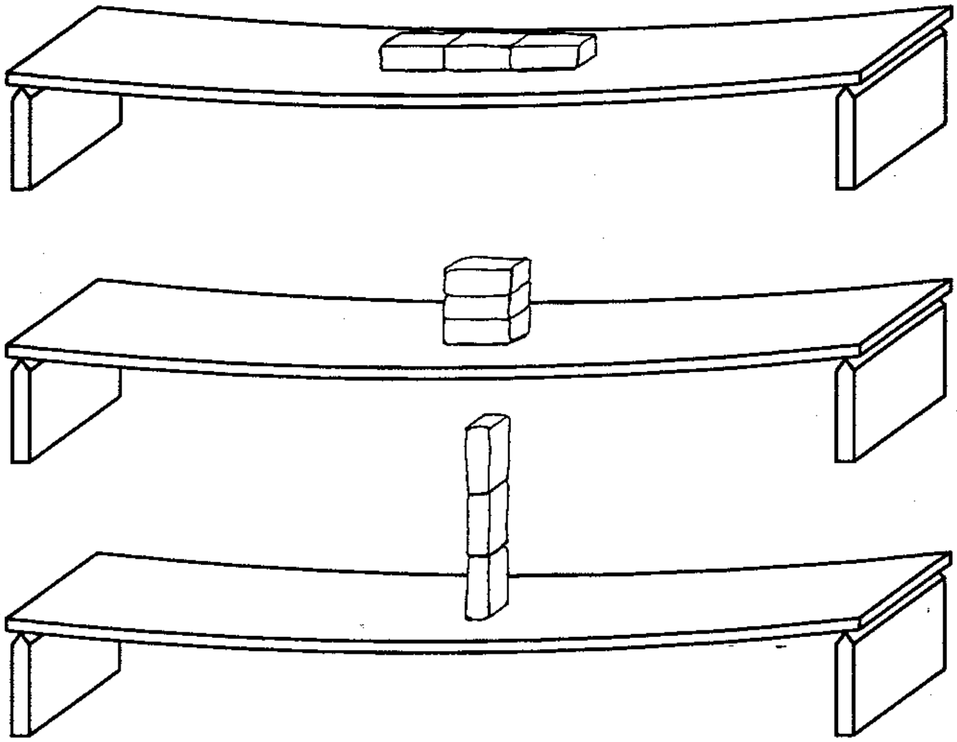
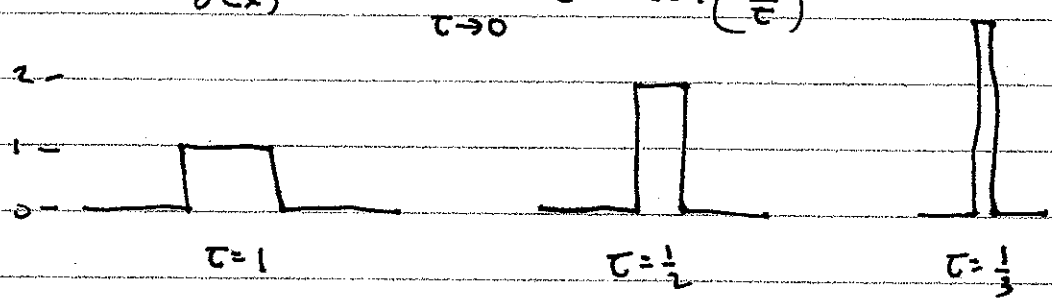


Figure 3-3 A sequence of pressure distributions suitable for defining a point load W. The applied pressure increases in theory *without limit* as we run through the sequence of taller stacks piled on smaller bases but the response of the plank approaches a limit.

As the bricks' force becomes more an more concentrated, we reach a limit in the board deflection. This suggests that we can describe the δ-function as a limit as follows:

$$\delta(x) = \lim_{\tau \to 0} \tau^{-1} \operatorname{rect}\left(\frac{x}{\tau}\right)$$



In each case the area of the function is unity, satisfying

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

Here we have used

$$\text{rect}(x) = \begin{cases} 1 & |x| < \frac{1}{2} \\ 0 & |x| > \frac{1}{2} \end{cases}$$

In two dimensions, we have

$${}^2\text{rect}(x,y) = \text{rect}(x)\,\text{rect}(y)$$

$$= \begin{cases} 1 & |x| < \frac{1}{2} \text{ and } |y| < \frac{1}{2} \\ 0 & \text{else} \end{cases}$$

Looking at our sequence for ${}^2\delta(x,y)$, then

$${}^2\delta(x,y) = \lim_{\tau \to 0} \tau^{-2}\, \text{rect}\left(\frac{x}{\tau}\right) \text{rect}\left(\frac{y}{\tau}\right)$$

$$= \lim_{\tau \to 0} \tau^{-2}\, {}^2\text{rect}\left(\frac{x}{\tau}, \frac{y}{\tau}\right)$$

In 1-D the sequence consisted of rectangles of unit area centered at the origin, in 2-D we have rectangular unit volumes sitting at the origin.

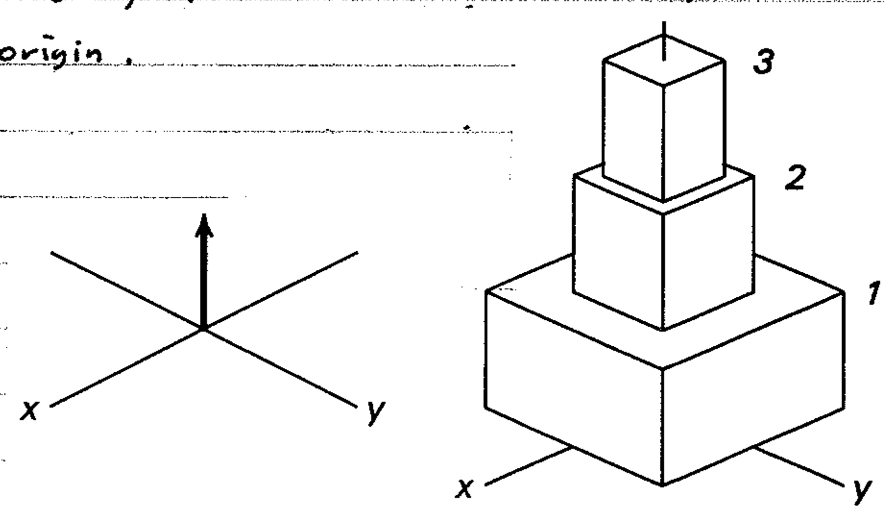


Figure 3-4 An arrow of unit height is the conventional representation of ${}^2\delta(x, y)$. The concept is defined by a sequence of functions 1, 2, 3, ... Note that the ${}^2\delta(x, y)$ is not the limit of the defining sequence which, at the interesting point (0, 0), has no limit.

So, we have the following recipe for evaluating the effects of two-dimensional impulses.

1. Replace ${}^2\delta(x,y)$ by $\tau^{-2}\,\mathrm{rect}\left(\frac{x}{\tau}, \frac{y}{\tau}\right)$

2. Evaluate our result to obtain an expression that depends on τ.

3. Calculate the limit as $\tau \to 0$

Note that we didn't have to use a rect function. A similar argument can be made, say, for a gaussian ($e^{-\pi x^2}$) function. Whichever form is easiest to evaluate should be used.

<u>Generalized functions</u>

While δ-functions were more slowly adopted in the mathematics community than by the physicists and engineers who needed them for their everyday work, by now there exists a more complete mathematical basis for them. Rigorous mathematical treatment is hampered by lack of continuity of the functions or any of their derivatives, or even of a substantiated procedure for integrating them. But by using the concept of sequences all of the mathematical issues can be dealt with, and we can usually obtain a close enough approximation with our rect function or Gaussian function sequences.

We'll now look at some of the impulse-related functions and their properties.

The shah functions

Shah functions in one and two-d, $III(x)$ and ${}^{2}III(x,y)$, arise naturally in sampling signals and also in representing periodic functions through convolution. In two-d, though, as in the properties of shape and curvature, the variety of phenomena is wider.

Sampling - results from multiplying a two-d function by a 2-D shah function. The points may be on a rectangular grid, or they may be regularly spaced in radius and angle. Sampling can also occur on line impulses in various configurations.

Periodicities of various types arise from convolution of the many shah functions with other functions.

Recall the one-d description of the shah function:

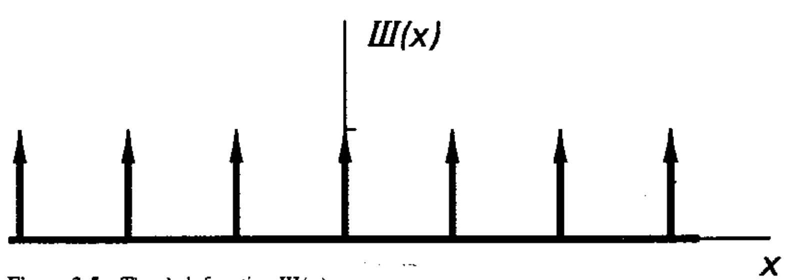


Figure 3-5 The shah function $III(x)$.

It is a small generalization to the 2-D analog:

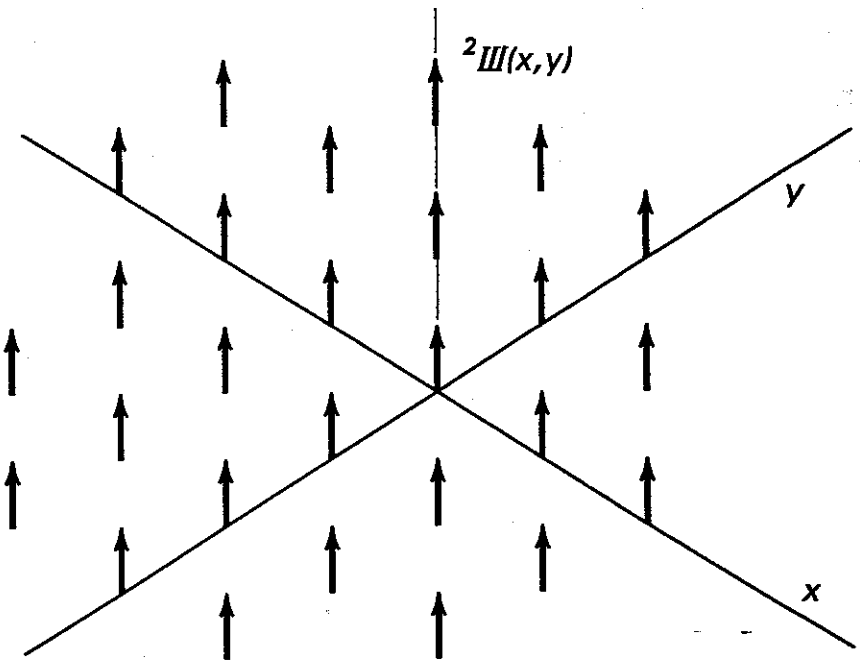


Figure 3-6 The bed-of-nails function or two-dimensional shah function $^2\text{III}(x, y)$.

In one dimension we defined

$$\text{III}(x) = \sum_{n=-\infty}^{\infty} \delta(x-n)$$

2-D version:

$$^2\text{III}(x,y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} {}^2\delta(x-m, y-n)$$

Sampling

In sampling, the product of a function $f(x,y)$ with ^2III:

sampled function = $f(x,y)\ {}^2\text{III}(x,y)$

Is a set of δ-functions (spikes) with "strengths" equal to the value of $f(x,y)$ at each location. If we want spacing at other than the integer values, say spaced X in x and Y in y, we use

$$\text{sampled function} = f(x,y) \frac{1}{|XY|}\ {}^2\text{III}\left(\frac{x}{X}, \frac{y}{Y}\right)$$

What does this expression mean?

Recall scaling rule $\delta(\frac{x}{a}) = |a|\,\delta(x)$, then it follows

$$^2\text{III}\left(\frac{x}{X}, \frac{y}{Y}\right) = |XY| \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - mX, y - nY)$$

Thus to represent an array of unit delta functions, we need to use the scaled version

$$\frac{1}{|XY|}\,^2\text{III}\left(\frac{x}{X}, \frac{y}{Y}\right)$$

Now, suppose we want a signal to be sampled and also limited in spatial extent, say for interpretation as a matrix in a computer. For example, we would like to represent an 8×8 matrix (64 values). If we index from 0 to 7 in each dimension multiplying by a suitable rect function does the trick:

$$^2\text{III}(x,y)\ ^2\text{rect}\left(\frac{x-3.5}{8}, \frac{y-3.5}{8}\right)$$

Line impulses

The generalizations of δ-functions to 2-D so far have been rather straight-forward, defining sequences in 2-D space analogously to sequences along a number line. But in two dimensions there are additional degrees of freedom which allow us to define function for which there is no correspondence in 1-D.

One such function is the line impulse, or line delta function. Here in influence of a quantity is concentrated along lines rather than at points. Examples:

line charges

illuminated slits

force on roof of building from its walls.

Straight-line impulse:

Consider a line charge confined to a single line in the plane:

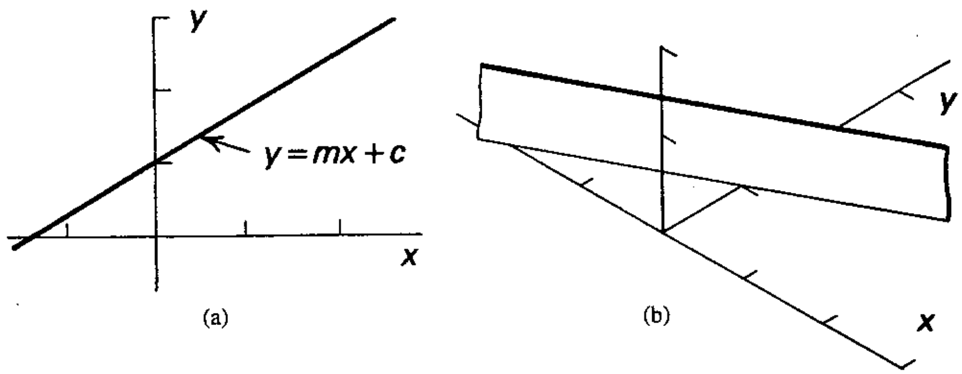


Figure 3-7 (a) A line charge $\delta(y - mx - c)$ on the (x, y)-plane lying on the line of slope m and y-intercept c. (b) A representation showing the line charge as a blade of height equal to the strength [in this case $\cos(\arctan m)$ or $1/\sqrt{1+m^2}$].

The charge is zero everywhere except along the line $y = mx + c$, suggesting the notation $\delta(y - mx - c)$.

Note that a 1-D δ-function suffices, as we have only a single argument to be set equal to zero. To understand this, consider a charge density $\sigma(x,y)$

$$\sigma(x,y) = K_0\, \delta(x)$$

What is the meaning of the coefficient K_0? $\sigma(x,y)$ is defined over the plane and equal to zero except along the line $x = 0$. What is its strength in coulombs per meter?

Consider a line of length L stretching from $(0,0)$ to $(0,L)$. To get the charge density, we can calculate the total charge and divide by L.

$$Q = \int_0^L \int_{-\infty}^{\infty} K_0\, \delta(x)\, dx\, dy$$

$$Q = \int_0^L k_o \, dy = k_o L$$

or $\frac{Q}{L} = k_o$. Thus we interpret k_o as the linear charge density in coulombs per meter.

Note: $\delta(x)$ has units of $\frac{1}{m}$, hence $k_o \delta(x)$ has units $C\,m^{-2}$, correct for a surface charge density $\sigma(x,y)$.

Thus, $\delta(x)$, interpreted as a function of both x and y, is a uniform line impulse on the y-axis, and has unit strength. Consequently, its integral over a unit length with be 1. If we think of a function $f(x,y)$ to have an integral as the volume situated on the area, $\delta(x)$ has unit "volume" per unit length.

Thus $\delta(y)$ is a unit strength line impulse on the x-axis. How about an arbitrary line? What would its strength be?

Consider the line impulse $\delta(y-mx-c)$. Clearly it is zero except along the line $y = mx+c$. So its location is readily determined, and now we need to evaluate its strength.

Recall our recipe for dealing with δ-functions:

1. Replace $\delta(x) = \tau^{-1} \operatorname{rect}\left(\frac{x}{\tau}\right)$
2. Calculate the quantity
3. Take limit as $\tau \to 0$

To determine the strength of the impulse, we calculate its ~~volume~~ integral along a unit length.

$$\delta(y-mx-c) \to \tau^{-1} \operatorname{rect}\left(\frac{y-mx-c}{\tau}\right)$$

Now $\text{rect}\left(\frac{y-mx-c}{\tau}\right) = \begin{cases} 1 & \left|\frac{y-mx-c}{\tau}\right| < \frac{1}{2} \\ 0 & \text{else} \end{cases}$

Let's plot the strip of width τ where the rect is non-zero:

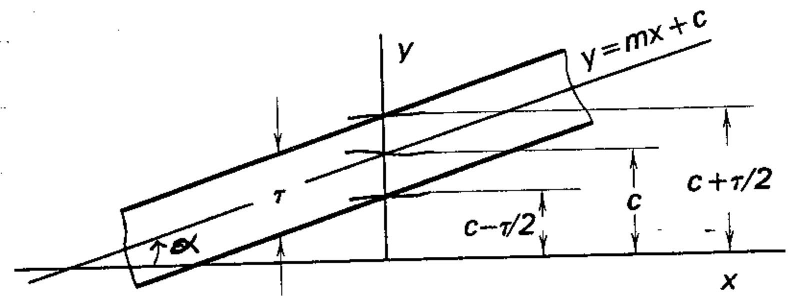


Figure 3-8 The two straight lines $y = mx + c \pm \tau/2$ are separated by a distance τ, measured *vertically* on the page.

The strip has width τ in the y-direction. If the angle of the line is $\alpha = \tan^{-1} m$, the width perpendicular to the strip is $\tau \cos\alpha$. Thus the cross section is a rectangle of size $\tau \cos\alpha$ by 1. Thus the integral giving the volume per unit length is simply the product of the height τ^{-1} and area $\tau \cos\alpha$, or $\cos\alpha$. The corresponding strength for a length L is $L\cos\alpha$. Since this result is independent of τ, in the limit the strength per unit length is still $\cos\alpha$.

In other words, the strength of the line delta is less than unity for lines at oblique angles to the axes. Our intuition is often misleading for δ-function notation, but we can always rely on the 3-step rule for evaluating the results correctly.