Functions of Two Variables

Can have both scalar and vector functions of 2 variables. As discussed previously, we will concentrate on scalar functions, recognizing that all the math we develop extends easily to vector functions where multiple pieces of information are available for each location in the image.

Surface shape

Much ~~infom~~ information is in the "shape" of a surface. This can refer to a physical surface, as in the case of topography, or an abstract surface such as the shape of a mathematical function.

Consider the following contour plot of an island:

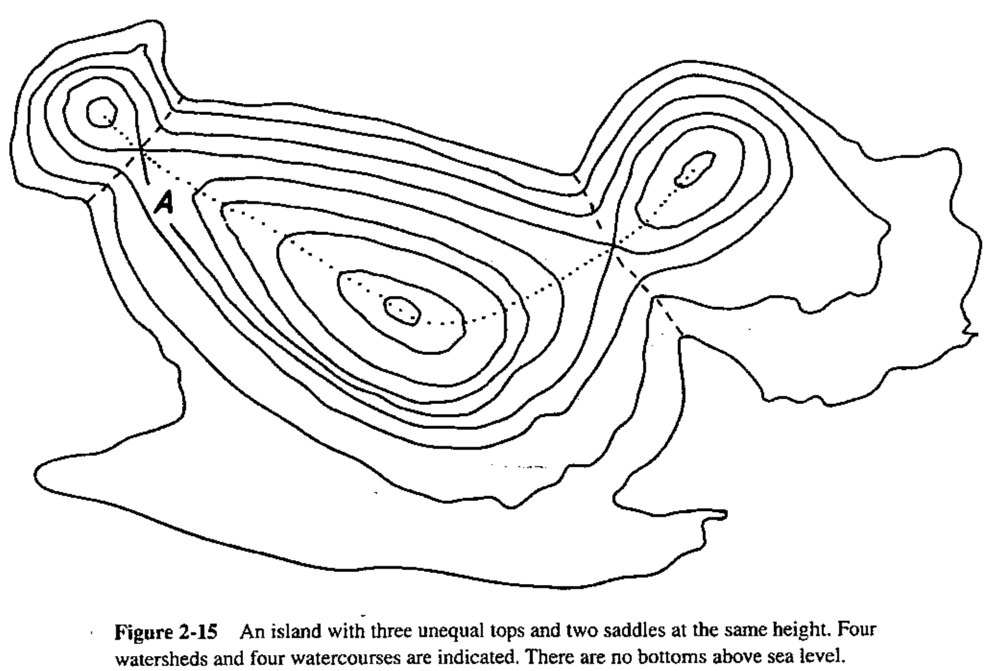


Figure 2-15 An island with three unequal tops and two saddles at the same height. Four watersheds and four watercourses are indicated. There are no bottoms above sea level.

This island has three peaks (maxima), no significant depressions (minima), and two saddle points.

We identify a saddle point as a spot where a contour line intersects itself.

Also we can infer slope information readily from the contour line spacing. Closely spaced contours imply that the height changes rapidly with position, a characteristic of steep slopes. Traversing along a contour line entails no change of elevation, or zero slope.

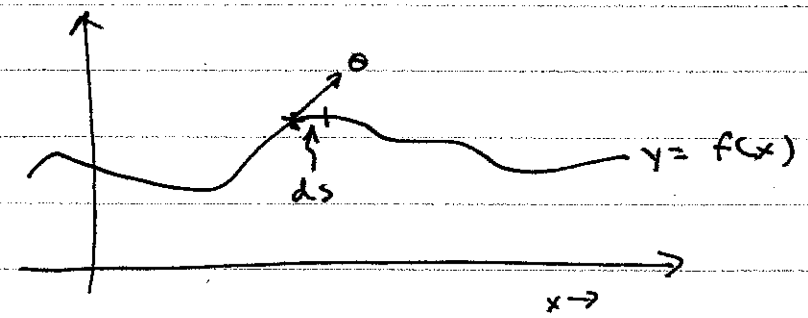
How might you approach the central peak of the island to have the easiest hike, assuming the terrain is passable everywhere?

For the more abstact case where we are not talking about physical topography, we still can think of a function $f(x,y)$ as a shaped surface. If $f(x,y)$ represents reflectance from a printed picture, we can imagine an island with low elevations where $f(x,y) \sim 0$ and high elevations where $f(x,y) \sim 1$. Then we can talk about the same kinds of peaks and valleys we use for actual topographic surfaces.

<u>Curvature</u>

Closely related to shape is the idea of curvature, which we use to express how the surface changes as we progress along it. We can formally define curvature as the rate of change of direction per unit arc length, so for a function in one dimension $y = f(x)$

$$\text{curvature} \equiv \frac{d\theta}{ds}$$



Note that the curvature is defined in terms of the distance along the curve, not with respect to x or y. Expressing in terms of the original variables we find

$$\text{curvature} = \frac{d^2y}{dx^2} \frac{1}{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}$$

or that curvature is approximately the second derivative of the function, and nearly so for small slopes ($\frac{dy}{dx} \approx 0$). So we see that curvature is the rate of change of slope.

Two dimensions impose a directionality on curvature: for a function $z = f(x,y)$

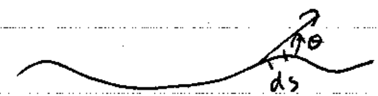
$$K_x = \frac{\partial^2 z}{\partial x^2} \frac{1}{\left(1+\left(\frac{\partial z}{\partial x}\right)^2\right)^{3/2}}$$

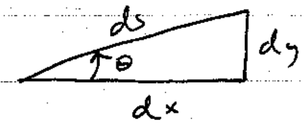
$$K_y = \frac{\partial^2 z}{\partial y^2} \frac{1}{\left(1+\left(\frac{\partial z}{\partial y}\right)^2\right)^{3/2}}$$

where K_i is the curvature in the i-direction.

Note on the derivation of curvature formula:

$$\text{curvature} = \frac{d\theta}{ds}$$





$$\tan\theta = \frac{dy}{dx} \qquad \cos\theta = \frac{dx}{ds} \qquad ds^2 = dx^2 + dy^2$$

Calculate $\frac{d\theta}{dx}$:

$$\frac{d}{dx}(\tan\theta) = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$\sec^2\theta \, \frac{d\theta}{dx} = \frac{d^2y}{dx^2}$$

$$\left(\frac{ds}{dx}\right)^2 \frac{d\theta}{dx} = \frac{d^2y}{dx^2}$$

$$\frac{d\theta}{dx} = \frac{d^2y}{dx^2} \cdot \left(\frac{dx}{ds}\right)^2$$

Calculate $\frac{ds}{dx}$: $\quad \frac{ds}{dx} = \frac{1}{\cos\theta}$

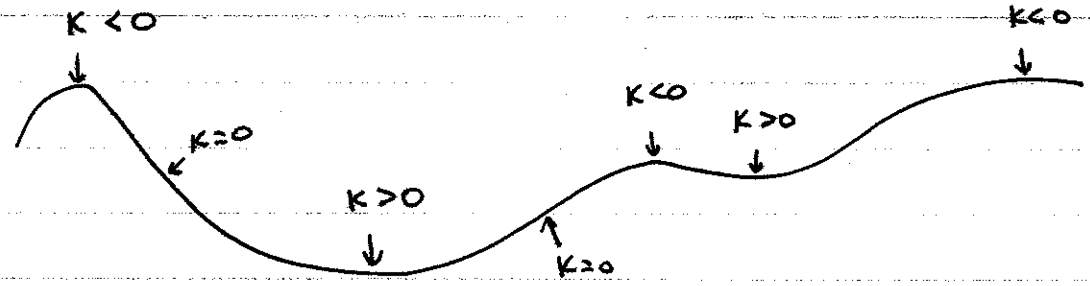
$$\frac{\frac{d\theta}{dx}}{\frac{ds}{dx}} = \frac{d\theta}{ds} = \frac{d^2y}{dx^2} \cdot \left(\frac{dx}{ds}\right)^2 \cdot \frac{dx}{ds}$$

$$= \frac{d^2y}{dx^2} \left(\frac{dx}{ds}\right)^3$$

$$= \frac{d^2y}{dx^2} \frac{(dx)^3}{\left((dx)^2 + (dy)^2\right)^{3/2}} = \frac{d^2y}{dx^2} \frac{\left((dx)^2\right)^{3/2}}{\left((dx)^2 + (dy)^2\right)^{3/2}}$$

$$= \frac{d^2y}{dx^2} \frac{1}{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2}}$$

What does curvature signify physically?



Peaks have negative curvature, troughs have positive curvature, sections of constant slope have zero curvature.

In two dimensions, where would:

Curvature be negative in x and y directions?

Curvature be positive in x and y directions?

Curvature be positive in some directions and negative in others?

Thus a display of curvature information might convey a good picture of underlying shape.

Projections

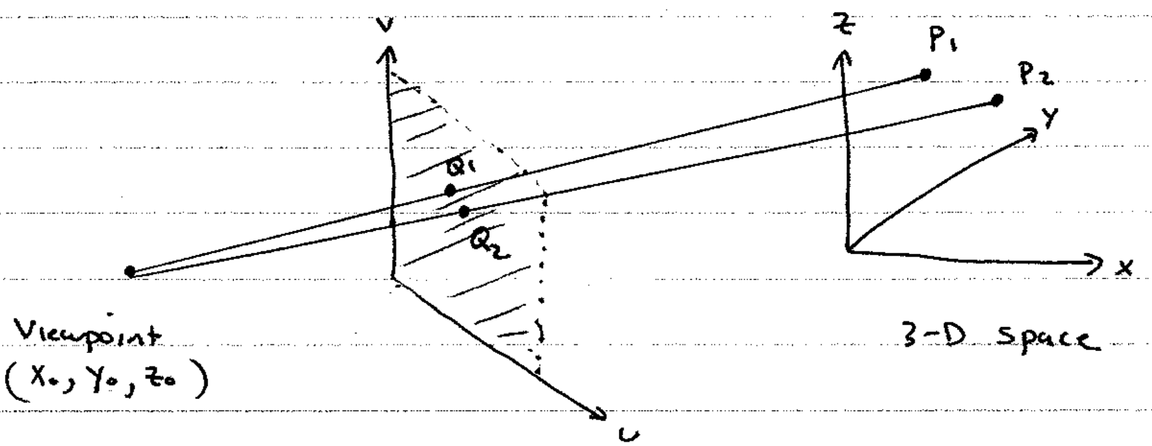
Representing a 3-D object on a 2-D screen or paper requires us to reduce the original object in some respect. In order to illustrate faithfully the desired properties of the original, 3-D distribution, some care must be taken in how to do the projection, because inevitably some information is lost.

We in fact come across projections in many forms every day. Any photograph represents a projection of one kind, where we use contextual clues such as one item being placed in front of another or the relative sizes of objects to infer the original 3-D view.

Another common projection is the one we see in conventional x-ray pictures. In an x-ray the integrated tissue density through a body is displayed, and the doctor's knowledge of how the internal organs are arranged provide the additional information needed to interpret the result.

Perspective projections

Perspective views result from choosing a viewpoint (x_0, y_0, z_0) in a 3-D space, and representing each point in the space on a plane, usually perpendicular to the look direction, where a ray from that point intersects the plane.



So points P_1 and P_2 in 3-D space map into points Q_1 and Q_2 in 2-D space. Varying the viewpoint obviously changes the projection dramatically.

We can derive the mathematical transformation of the coordinates of P_1 ~~2~~ and P_2 to Q_1 and Q_2 readily, especially if we adopt the following set of coordinates:

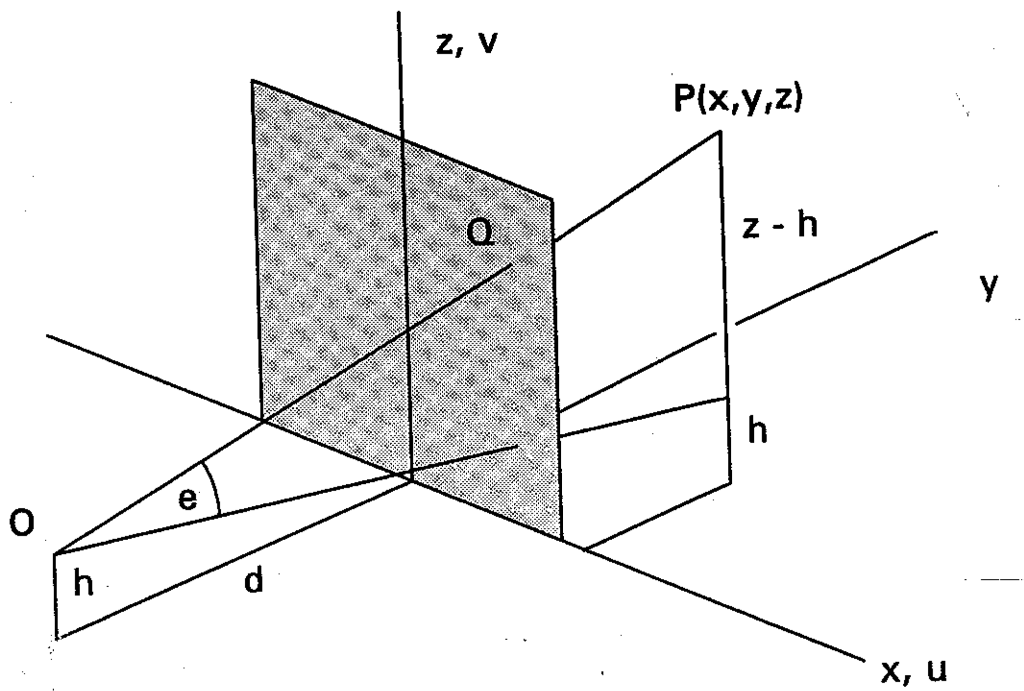


Figure 2-18 The relationship between the space coordinates (x, y, z) of the object point P and the picture-plane coordinates (u, v) of the projected point Q, when the (x, z)-plane is chosen to coincide with the (u, v)-plane.

The origin for the eye is located at $(0, -d, h)$; geometrical construction leads to

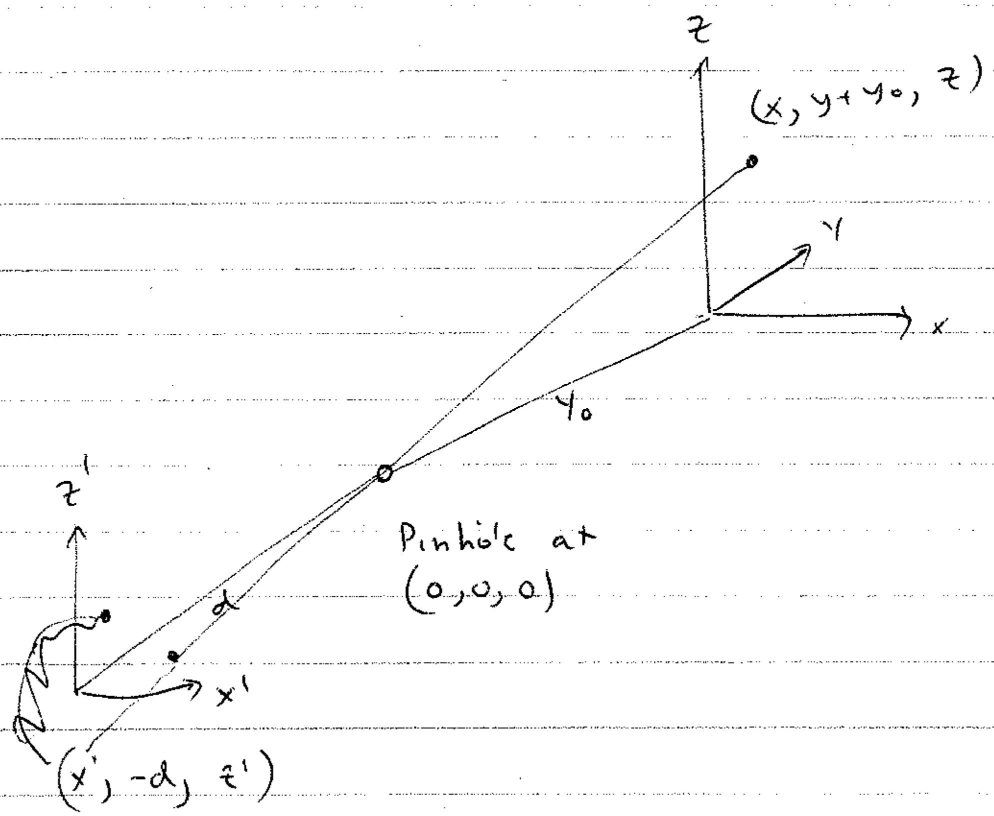
$$u = \frac{x d}{y+d}$$

$$v = h + \frac{(z-h) d}{y+d}$$

Note that the right hand sides of these equations involve three variables, x, y, and z, while the left hand sides only two, u and v. Thus three dimensions are reduced to two.

Is this operation invertible? What does that mean?

Pinhole camera



Vector from pinhole to $(x, y+y_0, z)$
Any point on that line is thus $a\begin{pmatrix}x\\y\\z\end{pmatrix}$ ← evaluate at $y=-d$

$$\begin{pmatrix} ax \\ a(y+y_0) \\ az \end{pmatrix} = \begin{pmatrix} x' \\ -d \\ z' \end{pmatrix}$$

$$\Rightarrow a = \frac{-d}{y+y_0}, \quad \left.\begin{aligned} x' &= \frac{-d}{y+y_0}\,x \\ z' &= \frac{-d}{y+y_0}\,z \end{aligned}\right\} \text{equations of output point}$$

<u>Inverse problems with perspective</u>

So it is straight-forward to determine how the 2-D perspective view of an object appears as a function of viewing geometry. Often we are faced with the <u>inverse problem</u>, where we are presented with one or more 2-D perspective views and asked to reconstruct the 3-D original object.

We have mentioned previously the use of contextual clues for qualitatively interpreting perspective views. We use many of these in our everyday experiences - name some of them:

Objects in foreground / background

Sizes

Lighting clues

⋮

How can some of these be made quantitative?

Orthographic projections:

One method of aiding quantitative interpretation is orthographic ~~reditions~~ renditions. An <u>orthographic</u> representation is one where the viewpoints are all an infinite distance away, and are all perpendicular to ~~each other~~ the object plane. These projections are indispensable for engineering and architectural drawings. Typically the three views used are of the front, side, and top of the item to be displayed.

A significant advantage of the orthographic projection is that all distances are true, that is given a single scale marker all dimensions may be inferred. This doesn't occur for many perspective drawings.

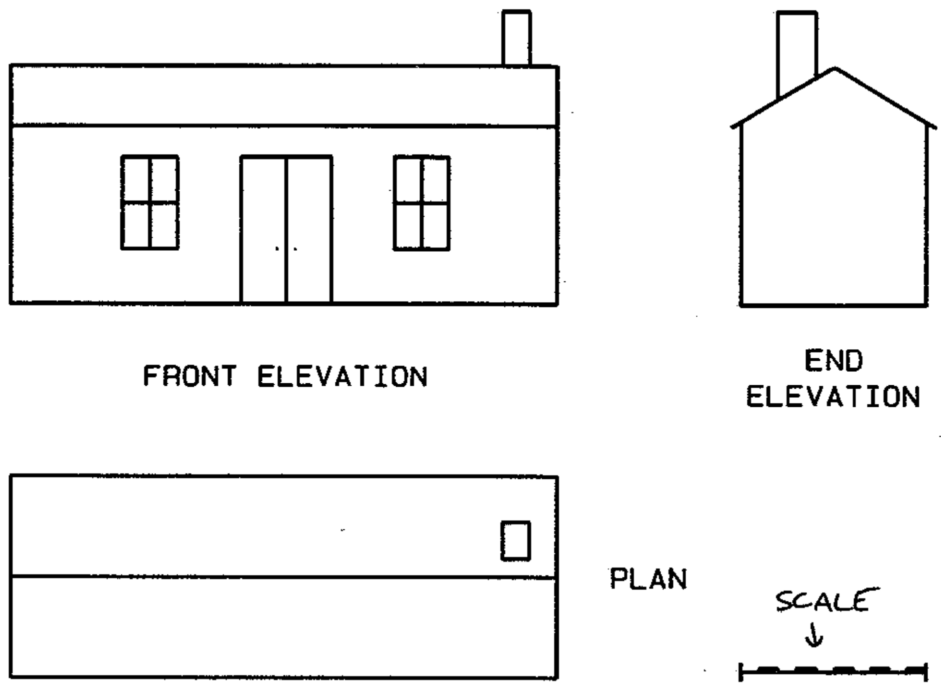


Figure 2-20 Orthographic projections in the form of plan, elevation, and side elevation.

Isometric projection

A certain orthographic projection, the isometric projection, is taken from a special direction so that the object's principal axes (x, y, and z) are all inclined equally. In other words, if $\hat{n}$ represents a vector in the viewing direction;

$$\hat{n}\cdot\hat{x} = \hat{n}\cdot\hat{y} = \hat{n}\cdot\hat{z} \qquad (*)$$

This turns out to be an angle of $\tan^{-1}\sqrt{2} = 54.7°$ between an axis and the look direction, and the angle between any axis and the picture plane is $35.3°$.

Because the projections in (*) are all equal, any given distance in the image along one of the axes projects equally. Thus for an engineering drawing distances in the cardinal directions are represented faithfully, and the object can be reproduced accurately. However, diagonal and other off-axis distances are not displayed accurately.

Thus the isometric projection contains less information than a full set of orthographic projections. However, for many (if not most) people it is easier to visualize the 3-D shape of an object in the isometric display. Once again we are led to the situation where thought must be paid to how the information will be used.

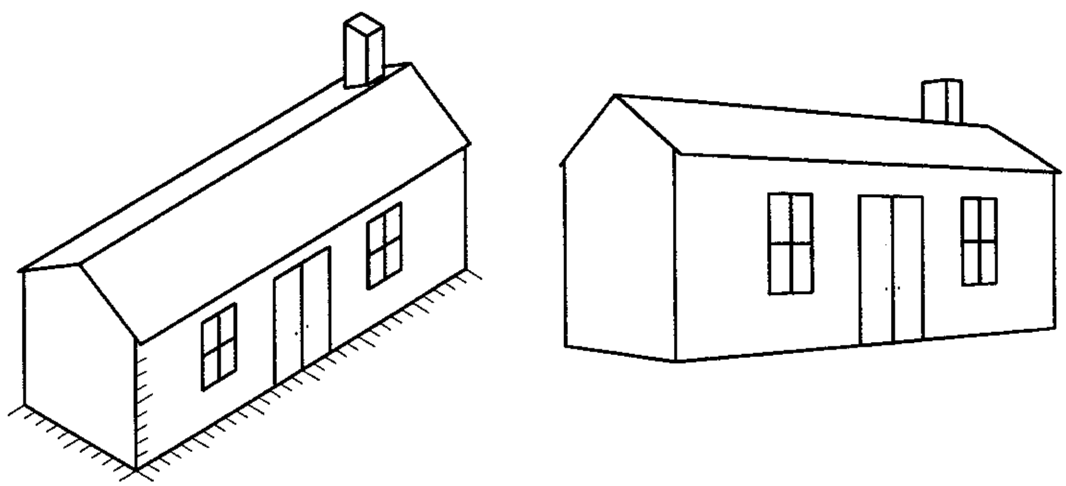


Figure 2-21 An isometric drawing (left), where correct distances can be measured to the same scale, in the three principal directions, and a perspective drawing (right), which looks natural but is not suited to measurement of dimensions.