EE241: Waves I **Autumn 2005–2006**

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Homework #3

Due Date: October 21, 2005, (Submit in class, or outside Packard 331 before 4:30 PM).

Reading Assignment:

"Reader"	Chapter 18, p.5ff
	Chapter 4
E&H	Section 4.7

Problems:

1. 'Slowness' [30 points]

   We can rewrite the harmonic solution in the form,

   $$\Psi = A\exp[-j\omega(\frac{x}{v_x} + \frac{y}{v_y} + \frac{z}{v_z} - t)]$$

   where we recognize that

   $$k_x = \frac{\omega}{v_x}, \text{etc.}$$

   Inverse speed, $\Lambda = v^{-1}$, is known as 'slowness.'

   With this, we can rewrite the wave expression, above, as

   $$\Psi = A\exp[-j\omega(x\Lambda_x + y\Lambda_y + z\Lambda_z - t)]$$

   where

   $$\Lambda^2 = \Lambda_x^2 + \Lambda_y^2 + \Lambda_z^2$$

   As with the wave vector **k**, the 'real' components of the slowness vector Λ determine the propagation direction and *phase* velocity of a harmonic wave ($\mathbf{k} = \omega\overline{\Lambda}$).

   Imaginary components of the slowness determine the orientation and relative strength of an exponential variation/distribution of the amplitude of the wave.

   (a) Examine the form of Ψ, and show that the amplitude is constant along lines in the direction of propagation, but can vary exponentially in planes perpendicular to the direction of propagation. For purposes of this problem consider only waves travelling parallel to the principal axes, if you wish. [5 points]

(b) Consider an instance in which the Λ_i are either pure real or pure imaginary. Take $\Lambda_y = 0$, Λ_x to be real. What are the forms of Λ_z for $\Lambda_x > \Lambda$, $\Lambda_x < \Lambda$? Use β to represent the attenuation coefficient in the z direction. [5 points]

(c) For the two conditions above what are the corresponding forms of $\Psi(x,z,t)$? (Waves that decay exponentially in a direction perpendicular to the propagation are referred to as "evanescent waves." We will revisit these in a few weeks.) [5 points]

(d) In a simple 'slowness diagram' we plot Λ_z (and β, when appropriate) vs. Λ_x, for $-\infty < \Lambda_x < +\infty$. Sketch this diagram, identifying regions corresponding to distinct forms of propagation, and describe the resulting curves in their generic form. Indicate geometrically on your sketch the lengths or distances representing Λ_x, Λ_z, Λ, and β. [5 points]

(e) In the portion of your diagram for which $\Lambda_x^2 < \Lambda^2$, what are the relationships among Λ_x, Λ_z, Λ, and the angle of propagation θ with respect to the x-axis? [5 points]

(f) Can you find the corresponding relationship in the region for which $\Lambda_x^2 > \Lambda^2$, in terms of an imaginary angle γ? [5 points]

2. Hamilton's principle [25 points]

(a) Use Hamilton's principle to find the equation of motion (in terms of the angular offset from vertical θ) of a pendulum with length l and end-mass m. Take the rod to be massless. Linearize the equation by assuming θ is small. [10 points]

(b) What is the natural frequency of the pendulum? [2 points]

(c) Express the Lagrangian of a simple pendulum in terms of Cartesian coordinates, using the transformation $x = l\cos\theta$ and $y = l\sin\theta$. Note that $x^2 + y^2 = l^2$. Rewrite the Lagrangian in terms of x and dx/dt. Obtain the corresponding dynamical equation and show that it is identical to that found in part b. [10 points]

(d) Comment on the similarities and differences between kinetic and potential energy storage in a pendulum and an acoustic wave. [3 points]

3. Coupled mass-spring oscillator [20 points]

(a) Write the Lagrangian and corresponding Euler-Lagrange equations for a *coupled* two-dimensional mass-spring oscillator. Assume the two springs are of length d and lie parallel to the x-axis and y-axis, having spring constants of K_x and K_y, respectively. Assume mass M is at the origin. [10 points]

(b) If it is assumed that displacements are small compared to d such that first-order binomial expansion can be used to approximate square roots, what are the resultant reduced forms for the Lagrangian and Euler-Lagrange equations? (Note: The reduced equations will still be coupled.) [10 points]

4. Group and Phase Velocity [25 points]

   (a) Phase velocity (v_p) and group velocity (v_g) are related to frequency and wave number by ω/k and $d\omega/dk$, respectively. Show that $v_g = v_p - \lambda \frac{dv_p}{d\lambda}$. [5 points]

   (b) Certain water waves of large amplitude have phase velocities given by $v_p = g/\omega$ where g is the acceleration due to gravity. Determine the ratio of group velocity to phase velocity for such a wave. [5 points]

   (c) If $v_p = A\omega^n$, where A and n are constants, show that $v_g = v_p/(1-n)$. For what values of n is the dispersion normal? Anomalous? [5 points]

   (d) There exist media in which the product of phase velocity and group velocity is a constant. If such a medium possesses normal dispersion, how does the phase velocity depend on wavelength? Sketch *all* possible ω-k diagrams. [10 points]

5. Non-Newtonian Fluids [15 points]

   Fluids that obey Hooke's Law $(p = -B\frac{\partial \xi}{\partial x})$ are called Newtonian. On the other hand, non-Newtonian fluids do not obey Hooke's law, i.e., stress is not directly proportional to strain. Suppose there exists a non-Newtonian fluid that satisfies the relation:

$$p = -B\frac{\partial \xi}{\partial x} - C\left(\frac{\partial \xi}{\partial x}\right)^3$$

   (a) Derive the dynamic equation for this non-Newtonian fluid. [10 points]

   (b) Find a relationship between frequency (ω) and displacement amplitude. You may assume that the fluid is nearly Newtonian. (What does this imply?) [5 points]