

Random Variables

Lecture Outline

- Several Random Variables and Independence
- Gaussian Random Variables
- Random Processes
- Stationarity, Mean, and Autocorrelation
- Wide Sense Stationary (WSS) Processes

1. Several Random Variables and Independence

- Let X and Y be defined on the same probability space.
- Their joint CDF is $F_{XY}(x, y) = P(X \leq x, Y \leq y)$
- Their joint pdf is $f_{XY}(x, y) = \frac{\partial^2 F_{XY}(x, y)}{\partial x \partial y}$, so $F_{XY}(x, y) = \int_{-\infty}^x \int_{-\infty}^y f_{XY}(\phi, \nu) d\phi d\nu$.
- Their conditional probability is $f_Y(y|X = x) = f_{XY}(x, y) / f_X(x)$.
- Joint random variables are independent if $f_{XY}(x, y) = f_X(x)f_Y(y)$
- The sum of independent random variables has a pdf equal to the convolution of the pdfs. For this sum, its characteristic function is the product of characteristic functions, the mean is the sum of the means, and the variance of the sum is the sum of the variances.

2. Gaussian Random Variables, Q and complementary error functions

- A common model for noise in communication systems.
- pdf defined in terms of mean and variance

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} e^{-[(x-\mu_X)^2/\sigma_X^2]}.$$

- The CDF cannot be found in closed form. Defined in terms of the error function and complementary error function. Specifically, $p(X \leq x) = .5 \left[1 + \operatorname{erf} \left(\frac{x-\mu_X}{\sqrt{2}\sigma_X} \right) \right]$, where $\operatorname{erf}(u) = \frac{2}{\sqrt{\pi}} \int_0^u e^{-z^2} dz = -\operatorname{erfc}(-u)$.
- Complementary error function defined as $\operatorname{erfc}(u) = 1 - \operatorname{erf}(u)$.
- Q function: for $X \sim \mathcal{N}(0, 1)$, define $Q(x) = p(X > x) = .5\operatorname{erfc}(x/\sqrt{2})$. Then for $Z \sim \mathcal{N}(\mu_Z, \sigma_Z^2)$, $p(Z > z) = Q((z - \mu_Z)/\sigma_Z)$.

3. Central Limit Theorem (CLT)

- Let X_i be independent and identically distributed (i.i.d.).
- Let $Y = \sum_{i=1}^n X_i$ and $Z = (Y - \mu_Y)/\sigma_Y$.
- The CLT states that the distribution of Z as $n \rightarrow \infty$ converges to a Gaussian RV with mean 0 and variance 1.

4. Random Processes

- A random process $X(t)$ is defined on a probability space $(\Omega, \mathcal{E}, P(\cdot))$.
- A random process $X(t)$ is a function mapping from the sample space Ω to a set of functions.
- Samples of $X(t)$ are joint random variables with joint CDF is $F_{X(t_0)X(t_1)\dots X(t_n)}(x_0, \dots, x_n) = P(X(t_0) \leq x_0, X(t_1) \leq x_1, \dots, X(t_n) \leq x_n)$

5. Stationarity

- A random process is (strictly) stationary if time shifts don't affect its probability.
- So for all T and all sets of sample times (t_0, \dots, t_n) , $P(X(t_0) \leq x_0, X(t_1) \leq x_1, \dots, X(t_n) \leq x_n) = P(X(t_0 + T) \leq x_0, X(t_1 + T) \leq x_1, \dots, X(t_n + T) \leq x_n)$.

6. Mean and Autocorrelation

- The mean of a random process is defined as $E[X(t)]$.
- Stationary random processes have constant mean: $E[X(t)] = \mu_X$.
- The autocorrelation of a random process is defined as $R_X(t_1, t_2) = E[X(t_1)X(t_2)]$.
- For stationary random processes the autocorrelation depends only on time difference of the samples: $R_X(t_1, t_2) = R_X(\tau = t_2 - t_1)$.
- Autocorrelation measures correlation between samples of the process taken at different times.

7. Wide Sense Stationary Process (WSS)

- A process is WSS if its expected power is finite, $E[X^2(t)] < \infty$, its mean is constant, $E[X(t)] = \mu_X$, and its autocorrelation depends only on the time difference of the samples, $R_X(t_1, t_2) = R_X(\tau = t_2 - t_1)$.
- Intuitively, WSS processes are stationary in their first and second moments.
- Stationary processes are WSS but not vice versa.

Main Points:

- Gaussian random variables are a common model for noise.
- Probability of Gaussian RVs evaluated using erf, erfc, and/or Q functions.
- CLT gives distribution for shifted, normalized sum of i.i.d. RVs as a $\mathcal{N}(0, 1)$ Gaussian RV.
- Random process $X(t)$ maps sample space Ω to a set of functions.
- Samples of $X(t)$ are joint RVs.
- A process is stationary if time shifts don't affect probability characteristics of the process.
- A WSS process has a constant mean and an autocorrelation that only depends on the time difference.