Lecture 7 - EE 179: Introduction to Communications - Spring 2009

Random Variables

Lecture Outline

- **Random Variables**
- **CDF and PDF**
- **Mean, Moments, Variance, and Characteristic Functions**
- **Several Random Variables and Independence**
- **Gaussian Random Variables**

1. **Random Variables**
   - Random variables are defined on a probability space $(\mathbf{S}, \mathcal{E}, \mathbf{P}(\cdot))$.
   - A random variable X is a function mapping from the the sample space Ω to a subset of the real line.
   - If the random variable X takes on only discrete values on the real line it is called a discrete random variable.
   - If the random variable X takes continuous values on the real line it is called a continuous random variable.
   - The cumulative distribution function (CDF) of a random variable is defined as $F_X(x) = P[X \leq x]$.
   - The CDF is derived from the probability space as $P[X \leq x] = P[X^{-1}(-\infty, x)]$.

2. **CDF Properties**
   - Properties of the CDF are based on properties of the underlying probability measure.
   - The CDF satisfies $0 \leq F_X(x) = P[X^{-1}(-\infty, x)] \leq 1$.
   - The CDF is nondecreasing: $F_X(x_1) \leq F_X(x_2)$ for $x_1 \leq x_2$. That is because $F_X(x_2) = P[X^{-1}(-\infty, x_2)] = P[X^{-1}(-\infty, x_1)] + P[X^{-1}(x_1, x_2)] \geq P[X^{-1}(-\infty, x_1)] = F_X(x_1)$.

3. **Probability Density Function (pdf)**
   - The derivative of the CDF is the probability density function (pdf), $f_X(x) = \frac{d}{dx} F_X(x)$.
   - The pdf defines the probability that X lies in a given range of values: $P(x_1 \leq X \leq x_2) = P(X \leq x_2) - P(X \leq x_1) = F_X(x_2) - F_x(x_1) = \int_{x_1}^{x_2} f_X(x)dx$.
   - Since $F_X(\infty) = 1$ and $F_X(-\infty) = 0$, the pdf integrates to 1: $\int_{-\infty}^{\infty} f_X(x)dx = 1$.

4. **Mean, Moments, Variance, and Characteristic Functions**
   - The mean or expected value of X is defined as $\mu_X = E[X] = \int_{-\infty}^{\infty} x f_X(x)dx$.
   - Similarly, the mean of a function of X is defined as $E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)dx$.
   - The nth moment of X is defined as $E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)dx$. The second moment $n = 2$ is called the mean-square value of X.
   - The variance of X is defined as $Var[X] = \sigma_X^2 = E[(X - \mu_X)^2]$. Expanding the square yields $\sigma_X^2 = E[X^2] - \mu_X^2$.

- The standard deviation of X, σ_X, is the square root of its variance.
- The characteristic function of X is defined as $\phi_X(\nu) = E[e^{j\nu X}] = \int_{-\infty}^{\infty} f_x(x)e^{j\nu x}dx = \mathcal{F}^{-1}(f_x(x))$. So the pdf and characteristic function are Fourier transform pairs.

5. **Several Random Variables and Independence**

   - Let X and Y be defined on the same probability space.
   - Thier joint CDF is $F_{XY}(x,y) = P(X \leq x, Y \leq y)$
   - Their joint pdf is $f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y}$, so $F_{XY}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(\phi,\nu)d\phi d\nu$.
   - Their conditional probability is $f_Y(y|X=x) = f_{XY}(x,y)/f_X(x)$.
   - Joint random variables are independent if $f_{XY}(x,y) = f_X(x)f_Y(y)$
   - The sum of independent random variables has a pdf equal to the convolution of the pdfs. For this sum, its characteristic function is the product of characteristic functions, the mean is the sum of the means, and the variance of the sum is the sum of the variances.

6. **Gaussian Random Variables, Q and complementary error functions**

   - A common model for noise in communication systems.
   - pdf defined in terms of mean and variance

   $$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} e^{-[(x-\mu_X)^2/\sigma_X^2]}.$$

   - The CDF cannot be found in closed form. Defined in terms of the error function and complementary error function. Specifically, $p(X \leq x) = .5\left[1 + \text{erf}\left(\frac{x-\mu_X}{\sqrt{2}\sigma_X}\right)\right]$, where $\text{erf}(u) = \frac{2}{\sqrt{\pi}}\int_0^u e^{-z^2}dz = -\text{erf}(-u)$.
   - Complementary error function defined as $\text{erfc}(u) = 1 - \text{erf}(u)$.
   - Q function: for $X \sim \mathcal{N}(0,1)$, define $Q(x) = p(X > x) = .5\text{erfc}(x/\sqrt{2})$. Then for $Z \sim \mathcal{N}(\mu_Z, \sigma_Z^2), p(Z > z) = Q((z-\mu_Z)/\sigma_Z)$.

7. **Central Limit Theorem (CLT)**

   - Let X_i be independent and identically distributed (i.i.d.).
   - Let $Y = \sum_{i=1}^{n} X_i$ and $Z = (Y - \mu_Y)/\sigma_Y$.
   - The CLT states that the distribution of Z as $n \to \infty$ converges to a Gaussian RV with mean 0 and variance 1.

Main Points:

- Random variables are functions defined on a probability space mapping from the sample space to the real line.
- The CDF and pdf of a random variable are derived from the underlying probability space.
- The mean of a random variable is its average value. The variance is the second moment minus the mean squared.
- A function of a RV is another RV with the same probability space. The characteristic function of a random variable is the Fourier Transform of its pdf.
- Several RVs have a joint pdf. The pdf of a sum of RVs is the convolution of pdfs.
- Gaussian RVs are a common model for noise.
- CLT gives distribution for shifted, normalized sum of i.i.d. RVs as a $\mathcal{N}(0,1)$ Gaussian RV.