**Lecture 6 - EE 179: Introduction to Communications - Spring 2009**

# Random Signals, Probability Theory and Random Variables.

## Lecture Outline

- **Random Signals**
- **Introduction to Probability**
- **Mathematics of Probability**
- **Properties of Probability**
- **Conditional Probability and Bayes Rule**
- **Independent Events**
- **Random Variables**

1. **Random Signals**
   - Random signals chosen from a set of possible realizations.
   - Random signals characterized by their average PSD.
   - Filtering and modulation of random signals has the same properties as for power signals.
   - To study random processes, we need to develop a mathematical theory for analysis, which begins with probability theory.
2. **Introduction to Probability**
   - Mathematical characterization for random events/experiments.
   - Events must be repeatable and have statistical regularity, i.e. a large number of experiments have regularity in outcome patterns.
   - We define the probability of a particular outcome $A$ as the average number of times the outcome is $A$, i.e. as as the number of times the outcome of the experiment is $A$ divided by $n$, in the limit of large $n$.
3. **Mathematics of Probability**
   - Random events are defined on a probability space consisting of a sample space $S$ of possible outcomes, a set of events $\mathcal{E}$ that are subsets of $\mathbf{S}$, and a probability measure $P[\cdot]$ on these subsets.
   - We abbreviate a probability space as the triple $(\mathbf{S}, \mathcal{E}, \mathbf{P}[\cdot])$.
   - The probability measure has 3 properties:
     (a) $P[\mathbf{S}] = \mathbf{1}$.
     (b) $0 \leq P[A] \leq 1$ for any event $A \in \mathcal{E}$
     (c) If $A$ and $B$ are mutually exclusive, i.e. their intersection is zero, then $P[A \cup B] = P[A] + P[B]$.

4. **Properties of Probability**
    - $P[A^c] = 1 - P[A]$
    - If $A_1 \cup A_2 \cup \ldots \cup A_N = \Omega$ and $A_i \cap A_j = \oslash$ then $\sum P[A_n] = 1$.
    - If $A \cap B \neq \oslash$ then $P[A \cup B] = P[A] + P[B] - P[A \cap B]$.

5. **Conditional Probability and Bayes Rule**
    - The outcome of 1 event can affect the outcome of another.
    - We define the conditional probability of events as $P[B|A] = P[A \cap B]/P[A]$ assuming $P[A] = 0$.
    - We can also write this as $P[A \cap B] = P[A|B]P[B] = P[B|A]P[A]$.

6. **Independent Events**
    - If $A$ and $B$ are independent, $P[B|A] = P[B]$ and $P[A|B] = P[A]$.
    - Equivalently, $P[A \cap B] = P[A]P[B]$.

7. **Random Variables**
    - Random variables are defined on a probability space $(\mathbf{S}, \mathcal{E}, \mathbf{P}(\cdot))$.
    - A random variable $X$ is a function mapping from the the sample space $\Omega$ to a subset of the real line.
    - If the random variable $X$ takes on only discrete values on the real line it is called a discrete random variable.
    - If the random variable $X$ takes continuous values on the real line it is called a continuous random variable.
    - The cumulative distribution function (CDF) of a random variable is defined as $F_X(x) = P[X \leq x]$.
    - The CDF is derived from the probability space as $P[X \leq x] = P[X^{-1}(-\infty, x)]$.

8. **CDF Properties**
    - Properties of the CDF are based on properties of the underlying probability measure.
    - The CDF satisfies $0 \leq F_X(x) = P[X^{-1}(-\infty, x)] \leq 1$.
    - The CDF is nondecreasing: $F_X(x_1) \leq F_X(x_2)$ for $x_1 \leq x_2$. That is because $F_X(x_2) = P[X^{-1}(-\infty, x_2)] = P[X^{-1}(-\infty, x_1)] + P[X^{-1}(x_1, x_2)] \geq P[X^{-1}(-\infty, x_1)] = F_X(x_1)$.

9. **Probability Density Function (pdf)**
    - The derivative of the CDF is the probability density function (pdf), $f_X(x) = \frac{d}{dx} F_X(x)$.
    - The pdf defines the probability that $X$ lies in a given range of values: $P(x_1 \leq X \leq x_2) = P(X \leq x_2) - P(X \leq x_1) = F_X(x_2) - F_x(x_1) = \int_{x_1}^{x_2} f_X(x) dx$.
    - Since $F_X(\infty) = 1$ and $F_X(-\infty) = 0$, the pdf integrates to 1: $\int_{-\infty}^{\infty} f_X(x) dx = 1$.

10. **Mean, Moments, Variance, and Characteristic Functions**
    - The mean or expected value of $X$ is defined as $\mu_X = E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$.

- Similarly, the mean of a function of $X$ is defined as $E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$.
- The $n$th moment of $X$ is defined as $E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x) dx$. The second moment $n = 2$ is called the mean-square value of $X$.
- The variance of $X$ is defined as $Var[X] = \sigma_X^2 = E[(X - \mu_X)^2]$. Expanding the square yields $\sigma_X^2 = E[X^2] - \mu_X^2$.
- The standard deviation of $X$, $\sigma_X$, is the square root of its variance.
- The Chebyshev inequality relates the mean and variance of $X$ as $P(|X - \mu_X| \geq \epsilon) \leq \sigma_X^2/\epsilon$.
- The characteristic function of $X$ is defined as $\phi_X(\nu) = E[e^{j\nu X}] = \int_{-\infty}^{\infty} f_x(x) e^{j\nu x} dx = \mathcal{F}^{-1}(f_x(x))$. So the pdf and characteristic function are Fourier transform pairs. This will become significant when we look at sums of random variables.

**Main Points:**

- Random signals common in communication systems. Need probability theory and random processes to characterize them.
- Probability theory allows us to characterize random events mathematically.
- Random events defines on a probability space, with events as subsets and a probability measure.
- Conditional probability characterizes the effect of one event on another.
- Events are independent if their joint probability equals their product.
- Random variables are functions defined on a probability space mapping from the sample space to the real line.
- The CDF and pdf of a random variable are derived from the underlying probability space.