Key FT Properties, Special Functions, Sampling, Filtering.

Lecture Outline

- **Key Fourier Transform Properties**
- **Special Functions**
- **Sampling**
- **Filtering**
- **Ideal Filters**

1. **Key Properties of Fourier Transforms**

   - *Time-Scaling:* $g(at) \Leftrightarrow \frac{1}{|a|}G(f/a)$
   - *Duality:* If $g(t) \Leftrightarrow G(f)$, then $G(t) \Leftrightarrow g(-f)$.
   - *Frequency Shifting (Modulation):* $e^{j2\pi f_c t}g(t) \Leftrightarrow G(f - f_c)$
   - *Multiplication in Time:*

   $$g_1(t)g_2(t) \Leftrightarrow \int_{-\infty}^{\infty} G_1(\lambda)G_2(f-\lambda)d\lambda.$$

   - *Convolution in Time:*

   $$\int_{-\infty}^{\infty} g_1(t)g_2(t-\tau)d\tau \Leftrightarrow G_1(f)G_2(f).$$

2. **Special Functions**

   - The dirac delta function is a key function in Fourier analysis, as it described the impulse response of a filter.
   - An exponential in time becomes a delta function in the frequency domain at the frequency of the exponential.
   - Sinusoids become two delta functions at the positive and negative frequencies of the sinusoid.
   - A delta function train in the time domain becomes a delta function train in the frequency domain.

3. **Sampling**

   - Sampling multiplies a signal by a train of equally-spaced delta functions
   - Fourier transform of a delta function train is another delta function train. So sampling in time convolves the signal in frequency with a train of delta functions.

- Spectrum of original signal is periodically repeated in frequency at the sampling rate.
- Nyquist Sampling Theorem: A signal that is bandlimited between $[-B, B]$ can be recovered from its samples taken every $.5/B$ seconds.

4. **Filtering**

- A filter is defined by its impulse response $h(t)$, the filter output to a delta function input.
- For any input $x(t)$, the filter output is $x(t) * h(t)$.
- In frequency domain, filter output to input $x(t)$ is $X(f)H(f)$. Easier to study filtering in the frequency domain.
- Communication channels with frequency response $H(f)$ introduce distortion. Can compensate for this distortion via equalization.

5. **Ideal Filters**

- And ideal low-pass filter has frequency response $H(f) = Ke^{-j2\pi f t_0}, |f| \leq B$, $H(f) = 0, |f| > B$. So its amplitude is constant and phase is linear over bandwidth B.
- An ideal low pass filter passes all positive and negative frequency components of a signal below a given cutoff bandwidth B.
- An ideal band pass filter passes all positive and negative frequency components of a signal between a lower and upper cutoff frequency B_1 and B_2.
- We use ideal filters to approximate implementations of filters in real systems.

Main Points:

- Time-scaling, duality, frequency shifting, multiplication and convolution are key Fourier transform properties.
- Delta function, exponentials, and sinusoids are key functions in Fourier analysis. Delta function train in time is a delta function train in frequency.
- Must sample at twice the signal bandwidth to recreate a signal from its samples.
- Easier to study filtering effects in the frequency domain.
- Communication channels act as filters, thereby distorting signals. Can compensate for distortion via equalization.
- Ideal filters are approximations to real filter implementations.