Properties of Continuous Time Fourier Series

In this handout we prove the various properties of Fourier series that were discussed in class. For all of these properties, we will use the notation $\Leftrightarrow$ to denote the transformation between a periodic signal and its Fourier series coefficients.

In the formulas below we assume the following Fourier series transform pairs:

$$x_p(t) \Leftrightarrow \{c_n\} \text{ and } v_p(t) \Leftrightarrow \{d_n\}.$$

1. Linearity:

$$Ax_p(t) + Bv_p(t) \Leftrightarrow \{Ac_n + Bd_n\}.$$

Proof:

$$\frac{1}{T_0}\int_{T_0} \left(Ax_p(t) + Bv_p(t)\right) e^{-j2\pi n f_0 t} dt = \frac{A}{T_0}\int_{T_0} x_p(t) e^{-j2\pi n f_0 t} dt + \frac{B}{T_0}\int_{T_0} v_p(t) e^{-j2\pi n f_0 t} dt = Ac_n + Bd_n.$$

2. Multiplication/Modulation:

$$x_p(t)v_p(t) \Leftrightarrow \{c'_n = \sum_{k=-\infty}^{\infty} c_k d_{n-k}\}$$

.

The right hand side is a *discrete convolution*

Proof: By multiplying the Fourier series representations of $x_p(t)$ and $v_p(t)$ we get

$$\begin{aligned} z_p(t) &= \left(\sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t}\right)\left(\sum_{m=-\infty}^{\infty} d_m e^{j2\pi m f_0 t}\right) \\ &= \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} d_m e^{j2\pi m f_0 t} \\ &= \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} c_k d_m e^{j2\pi (k+m) f_0 t}. \end{aligned} \qquad (1)$$

Change variables: $n = m + k$ so that $m = n - k$ and the above becomes

$$\sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} c_k d_{n-k} e^{j2\pi n f_0 T} = \sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} c_k d_{n-k}\right) e^{j2\pi n f_0 T}$$

which has the desired form with

$$c'_n = \sum_{k=-\infty}^{\infty} c_k d_{n-k}.$$

3. Time Shifting:

$$x_p(t-\tau) \Leftrightarrow \{e^{-j2\pi n f_0 \tau} c_n\}.$$

Proof:

$$x_p(t-\tau) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 (t-\tau)} = \sum_{n=-\infty}^{\infty} \left(e^{-j2\pi n f_0 \tau} c_n\right) e^{j2\pi n f_0 t}$$

.

4. Time Reversal:

$$x_p(-t) \Leftrightarrow \{c_{-n}\}.$$

Proof:

$$x_p(-t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 (-t)} = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi(-n) f_0 t} = \sum_{m=\infty}^{-\infty} c_{-m} e^{j2\pi m f_0 t} = \sum_{m=-\infty}^{\infty} c_{-m} e^{j2\pi m f_0 t},$$

where the third equality follows from a change of variables and the last equality follows from a reordering of the summation.

5. Time Scaling:

$$x_p(at) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 (at)}.$$

Proof: Direct substitution. Note that the Fourier series coefficients are the same, but the exponential basis functions have changed (i.e. their time has been scaled by a).

6. Conjugation:

$$x_p^*(t) \Leftrightarrow \{c_{-n}^*\}.$$

Proof:

$$x_p^*(t) = \left(\sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t}\right)^* = \sum_{n=-\infty}^{\infty} c_n^* e^{-j2\pi n f_0 t} = \sum_{m=\infty}^{-\infty} c_{-m}^* e^{j2\pi m f_0 t} = \sum_{m=-\infty}^{\infty} c_{-m}^* e^{j2\pi m f_0 t},$$

where the third equality follows from a change of variables ($m = -n$).

7. Parseval's Relation:

$$\frac{1}{T_0}\int_{T_0} |x_p(t)|^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

Proof:

$$\begin{aligned}
\frac{1}{T_0}\int_{T_0} x_p(t) x_p^*(t)\, dt &= \frac{1}{T_0}\int_{T_0} x_p(t) \left(\sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t}\right)^* \\
&= \sum_{n=-\infty}^{\infty} c_n^* \underbrace{\frac{1}{T_0}\int_{T_0} x_p(t) e^{-j2\pi n f_0 t}\, dt}_{c_n} = \sum_{n=-\infty}^{\infty} |c_n|^2
\end{aligned}$$

Note we only replaced one of the signals by its Fourier series, replacing both would have meant more work!