Lecture 4: Review of Signals and Systems, Frequency Domain

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Signals and Systems: Frequency

Today:

- The Fourier transform in $2\pi f$
- Some important Fourier transforms
- Some important Fourier transform theorems
- Hilbert transform and analytic signals

Next Time:

- Communication channels and equalization
- Amplitude modulation
- Modulators

Based on Notes from John Gill

Fourier Transform: EE 102A

- In EE 102A, the Fourier transform was

$$G(j\omega) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt$$

  and the inverse transform was

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(j\omega) e^{j\omega t} d\omega$$

- We used $j\omega$ to make all of the transforms to be similar in form (Fourier, Laplace, DTFT, z)

Fourier Transform Definition

- In this class we will use $2\pi f$, instead of ω. If we replace ω with $2\pi f$, the Fourier transform is
$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}\,dt\,.$$
Noting that $d\omega = 2\pi\;df$, the inverse transform is then
$$g(t) = \int_{-\infty}^{\infty} G(f)e^{j2\pi ft}\,df\,.$$
- Note that these are almost completely symmetric. Only the sign of the complex exponential changes.
- This will simplify a lot of the transforms and theorems.

Fourier Transform Existence

If $g(t)$ is absolutely integrable, i.e.,

$$\int_{-\infty}^{\infty} |g(t)|\, dt < \infty$$

then $G(f)$ exists for every frequency f and is continuous.

If $g(t)$ has finite energy, i.e.,

$$\int_{-\infty}^{\infty} |g(t)|^2\, dt < \infty$$

then $G(f)$ exists for "most" frequencies f and has finite energy.

If $g(t)$ is periodic and has a Fourier series, then

$$G(f) = \sum_{n=-\infty}^{\infty} G(nf_0)\delta(f - nf_0)$$

is a weighted sum of impulses in frequency domain.

Fourier Transform Example in $2\pi f$

One-sided exponential decay is defined by $e^{-at}u(t)$ with $a > 0$:

$$g(t) = \begin{cases} 0 & t < 0 \\ e^{-at} & t > 0 \end{cases}$$

The Fourier transform of one-sided decay is (simple calculus):

$$\begin{aligned} G(f) &= \int_0^\infty e^{-j2\pi ft} e^{-at}\, dt = \int_0^\infty e^{-(a+j2\pi f)t}\, dt \\ &= \left[\frac{e^{-(a+j2\pi f)t}}{-(a+j2\pi f)} \right]_{t=0}^{t=\infty} = \frac{1}{a+j2\pi f} \end{aligned}$$

Compare this to the EE 102A transform

$$G(j\omega) = \frac{1}{a+j\omega}$$

Often you can easily just substitute $2\pi f$ for ω.

Fourier Transform Examples: Real Valued Signals

Many of the signals we will be dealing with are real, such as sound and video.
In this case the Fourier transform is conjugate symmetric

$$
\begin{aligned}
G^*(f) &= \Big(\int_{-\infty}^{\infty} g(t) e^{-j2\pi f t}\, dt \Big)^* = \int_{-\infty}^{\infty} g(t)^* (e^{-j2\pi f t})^*\, dt \\
&= \int_{-\infty}^{\infty} g(t) e^{j2\pi f t}\, dt = \int_{-\infty}^{\infty} g(t) e^{-j2\pi(-f)t}\, dt = G(-f)
\end{aligned}
$$

This means that the real part of the signal is *even* and the imaginary part is *odd*. This is called *Hermitian symmetry*.

This is important because we need only positive frequencies for real-valued signals.

We will exploit this often!

Duality

Fourier inversion theorem: if $g(t)$ has a Fourier transform $G(f)$,

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}\,dt$$
$$g(t) = \int_{-\infty}^{\infty} G(f)e^{j2\pi ft}\,df$$

The inverse transform differs from the forward transform only in the sign of the exponent.

If we consider $G(f)$ to be a function of t instead of f, and apply the Fourier Transform again

$$\int_{-\infty}^{\infty} G(t)e^{-j2\pi ft}\,dt = \int_{-\infty}^{\infty} G(t)e^{j2\pi(-f)t}\,dt = g(-f)$$

We can summarize this as

$$\mathcal{F}\{g(t)\} = G(f) \;\Rightarrow\; \mathcal{F}\{G(t)\} = g(-f)$$

This is the *principle of duality*. This is much easier to use with $2\pi f$.

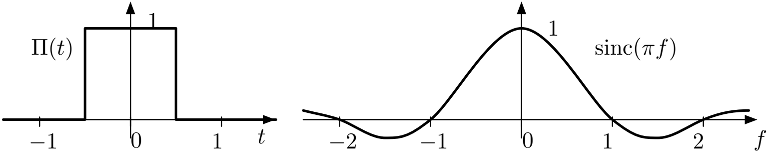
Important Fourier Transforms

The unit rectangle function $\Pi(t)$ is defined by

$$\Pi(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & |t| > \frac{1}{2} \end{cases}$$

Its Fourier transform is

$$\mathcal{F}\{\Pi(t)\} = \int_{-\infty}^{\infty} e^{-i2\pi f t}\Pi(t)\,dt = \int_{-1/2}^{1/2} e^{-i2\pi f t}\,dt = \frac{\sin \pi f}{\pi f} = \mathrm{sinc}(\pi f)$$



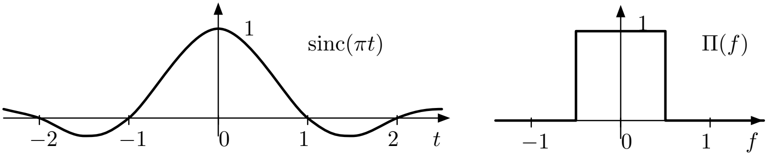
$$\Pi(t) \rightleftharpoons \mathrm{sinc}(\pi f)$$

Fact: every finite width pulse has a transform with unbounded frequencies.

By duality, we can also find

$$\mathrm{sinc}(\pi t) \rightleftharpoons \Pi(f)$$

since $\Pi(f)$ is even, so $\Pi(f) = \Pi(-f)$.



Fourier Transform Time Scaling

If $a > 0$ and $g(t)$ is a signal with Fourier transform $G(f)$, then

$$\mathcal{F}\{g(at)\} = \int_{-\infty}^{\infty} g(at) e^{-j2\pi ft} dt = \int_{-\infty}^{\infty} g(u) e^{-j2\pi fu/a} (du/a) = \frac{1}{a} G\Big(\frac{f}{a}\Big)$$

If $a < 0$, change of variables requires reversing limits of integration:

$$\mathcal{F}\{g(at)\} = -\frac{1}{a} G\Big(\frac{f}{a}\Big)$$

Combining both cases:

$$\mathcal{F}\{g(at)\} = \frac{1}{|a|} G\Big(\frac{f}{a}\Big)$$

Special case: $a = -1$. The Fourier transform of $g(-t)$ is $G(-f)$.

Compressing in time corresponds to expansion in frequency (and reduction in amplitude) and vice versa.

The sharper the pulse the wider the spectrum.

Fourier Transform Time Scaling Example

The transform of a narrow rectangular pulse of area 1 is

$$\mathcal{F}\left\{\frac{1}{\tau}\Pi(t/\tau)\right\} = \operatorname{sinc}(\pi\tau f)$$

In the limit, the pulse is the unit impulse, and its tranform is the constant 1.

We can find the Fourier transform directly:

$$\mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi f t}\, dt = e^{-j2\pi f t}\Big|_{t=0} = 1$$

so

$$\delta(t) \rightleftharpoons 1$$

The impulse is the mathematical abstraction of signal whose Fourier transform has magnitude 1 and phase 0 for all frequencies.

By duality, $\mathcal{F}\{1\} = \delta(f)$. All DC, no oscillation.

$$1 \rightleftharpoons \delta(f)$$

Important Fourier Transforms (cont.)

- Shifted impulse $\delta(t - t_0)$:

$$\mathcal{F}\{\delta(t-t_0)\} = \int_{-\infty}^{\infty} \delta(t-t_0) e^{-j2\pi f t}\, dt = e^{-j2\pi f t_0}$$

  This is a complex exponential in frequency.

$$\delta(t-t_0) \rightleftharpoons e^{-j2\pi f t_0}$$

  Then by duality

$$e^{j2\pi f_0 t} \rightleftharpoons \delta(f - f_0)$$

- Sinuoids:

$$\mathcal{F}\{\cos 2\pi f_0 t\} = \mathcal{F}\{\tfrac{1}{2}(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t})\} = \tfrac{1}{2}\delta(f-f_0) + \tfrac{1}{2}\delta(f+f_0)$$

$$\mathcal{F}\{\sin 2\pi f_0 t\} = \mathcal{F}\{\tfrac{1}{2i}(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t})\} = \tfrac{1}{2i}\delta(f-f_0) - \tfrac{1}{2i}\delta(f+f_0)$$

Important Fourier Transforms (cont.)

- Laplacian pulse $g(t) = e^{-a|t|}$ where $a > 0$. Since

$$g(t) = e^{-at}u(t) + e^{at}u(-t)\,,$$

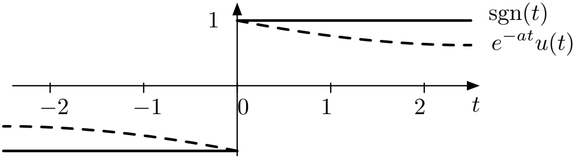
we can use reversal and additivity:

$$G(f) = \frac{1}{a + j2\pi f} + \frac{1}{a - j2\pi f} = \frac{2a}{a^2 + 4\pi^2 f^2}\,.$$

This is twice the real part of the Fourier transform of $e^{-at}u(t)$

- The signum function can be approximated as

$$\mathrm{sgn}(t) = \lim_{a\to 0}\left(e^{-at}u(t) - e^{at}u(-t)\right)$$



- This has the Fourier transform

$$\begin{aligned}\mathcal{F}\{\text{sgn}(t)\} &= \lim_{a\to 0}\left(\frac{1}{a+j2\pi f}-\frac{1}{a-j2\pi f}\right) \\ &= \frac{1}{j\pi f}\end{aligned}$$

- The unit step function is

$$u(t) = \frac{1}{2} + \frac{1}{2}\text{sgn}(t)$$

This has the Fourier transform

$$\mathcal{F}\{u(t)\} = \tfrac{1}{2}\delta(f) + \frac{1}{j2\pi f}$$

Hilbert Transform

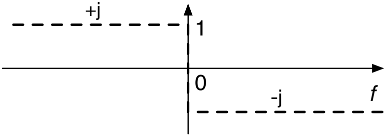
An import filter has an impulse response

$$h(t) = \frac{1}{\pi t}$$

By duality, using the $\mathrm{sgn}(t)$ transform we found above,

$$H(f) = -j \ \mathrm{sgn}(f)$$

which looks like this

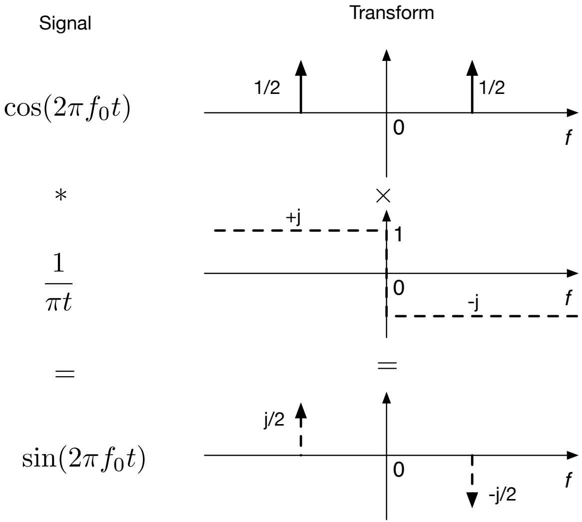


To see why this is important, consider

$$\cos(2\pi f_0 t) * \frac{1}{\pi t}$$

What does this do?

Hilbert Transform



It has turned a cosine into a sine! This will turn up frequently.

Analytic Signal

The dual of the unit step is also useful

$$H(f) = \frac{1}{2}(1 + \operatorname{sgn}(f))$$

By duality this corresponds to the impulse response

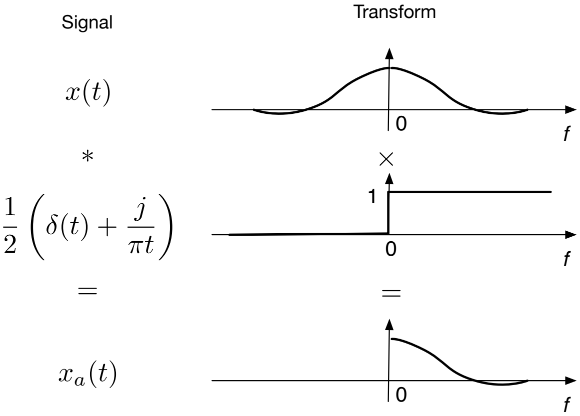
$$h(t) = \frac{1}{2}\left(\delta(t) + \frac{j}{\pi t}\right)$$

Consider what happens when we take a real signal, and convolve it with $h(t)$

$$x(t) * h(t)$$

What does this do?

Analytic Signal



This is a signal that just has positive frequencies!

We will see this when we talk about signal sideband transmission.

Fourier Transform Properties

- Time delay causes linear phase shift.

$$\begin{aligned}\mathcal{F}\left\{g(t-t_0)\right\} &= \int_{-\infty}^{\infty} g(t-t_0)e^{-j2\pi ft}dt = \int_{-\infty}^{\infty} g(u)e^{-j2\pi f(u+t_0)}du \\ &= e^{-j2\pi ft_0}\int_{-\infty}^{\infty} g(u)e^{-j2\pi fu}du = e^{-j2\pi ft_0}G(f)\end{aligned}$$

So,

$$g(t-t_0) \rightleftharpoons e^{-j2\pi ft_0}G(f)$$

- Then, by duality, we get frequency shifting (modulation):

$$e^{j2\pi f_c t}g(t) \rightleftharpoons G(f-f_c)$$

$$\cos(2\pi f_c t)g(t) \rightleftharpoons \frac{1}{2}G(f+f_c) + \frac{1}{2}G(f-f_c)$$

Fourier Transform Properties (cont.)

- Convolution in time. The convolution of two signals is

$$g_1(t) * g_2(t) = \int_{-\infty}^{\infty} g_1(u) g_2(t-u)\, du$$

- The Fourier transform of the convolution is the product of the transforms.

$$g_1(t) * g_2(t) \rightleftharpoons G_1(f) G_2(f)$$

  The Fourier transform reduces convolution to a simpler operation.

Fourier Transform Properties (cont.)

- Multiplication in time.

$$g_1(t)g_2(t) \rightleftharpoons \int_{-\infty}^{\infty} G_1(\lambda)G_2(f-\lambda)\,d\lambda$$

- There is no $\frac{1}{2\pi}$ factor, as there was in EE 102A.
- Convolution in one domain goes exactly to multiplication in the other domain, and multiplication to convolution.
- The modulation theorem is a special case.

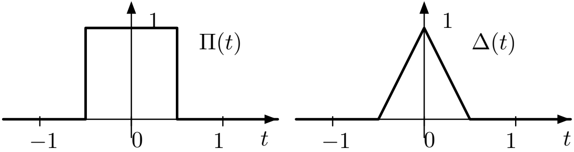
$$\begin{aligned}\mathcal{F}\left\{g(t)\cos(2\pi f_c t)\right\} &= \mathcal{F}\left\{g(t))\right\} * \mathcal{F}\left\{\cos(2\pi f_c t)\right\} \\ &= G(f) * \left(\tfrac{1}{2}\delta(f+f_0) + \tfrac{1}{2}\delta(f-f_0)\right) \\ &= \tfrac{1}{2}G(f+f_c) + \tfrac{1}{2}G(f-f_c)\end{aligned}$$

The triangle function $\Delta(t)$ and its Fourier transform

- The book defines the triangle function $\Delta(t)$ as

$$\Delta(t) = \begin{cases} 1 - 2|x| & |x| \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

This is another unfortunate choice, but not as bad as $\mathrm{sinc}(t)$!



- The triangle function can be written as twice the convolution of two rectangle functions of width $\frac{1}{2}$.

$$\Delta(t) = 2\ \Pi(2t) * \Pi(2t)$$

where the factor of 2 is needed to make the convolution 1 at $t = 0$.

- The Fourier transform is then

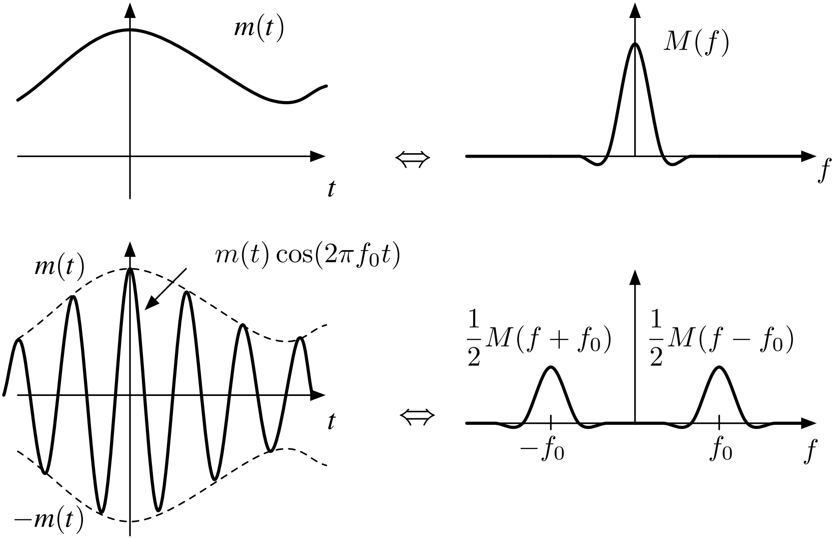
$$\begin{aligned}\mathcal{F}\left\{\Delta(t)\right\} &= \mathcal{F}\left\{2\Pi(2t) * \Pi(2t)\right\} \\ &= 2\mathcal{F}\left\{\Pi(2t)\right\}^2 \\ &= 2\left(\frac{1}{2}\mathrm{sinc}\left(\frac{\pi}{2}f\right)\right)^2 \\ &= \frac{1}{2}\mathrm{sinc}^2\left(\frac{\pi}{2}f\right)\end{aligned}$$

Then

$$\Delta(t) \rightleftharpoons \frac{1}{2}\mathrm{sinc}^2\left(\frac{\pi}{2}f\right)$$

Modulation Theorem

Modulation of a baseband signal creates replicas at $\pm$ the modulation frequency



Applications of Modulation

- For transmission by radio, antenna size is proportional to wavelength.

  Low frequency signals (voice, music) must be converted to higher frequency.

- To share bandwidth, signals are modulated by different carrier frequencies.
  - North America AM radio band: 535–1605 KHz (10 KHz bands)
  - North America FM radio band: 88–108 MHz (200 KHz bands)
  - North America TV bands: VHF 54–72, 76–88, 174–216, UHF 470–806, 806–890

  Frequencies can be reused in different geographical areas.

  With digital TV, channel numbers do not correspond to frequencies.

Rats laughing: `http://www.youtube.com/watch?v=j-admRGFVNM`

Bandpass Signals

Bandlimited signal: $G(f) = 0$ if $|f| > B$.

Every sinusoid $\sin(2\pi f_c t)$ has bandwidth f_c.

If $g_c(t)$ and $g_s(t)$ are bandlimited, then

$$m(t) = g_c(t)\cos(2\pi f_c t) + g_s(t)\sin(2\pi f_c t)$$

is a *bandpass* signal. Its Fourier transform or spectrum is restricted to

$$f_c - B < |f| < f_c + B$$

The bandwidth is $(f_c + B) - (f_c - B) = 2B$.

Most signals of interest in communications will be either bandpass (RF), or baseband (ethernet).

Next time

- Lab this Friday : Finding and decoding airband AM
- Next class Monday : 3.6 – 3.8 in Lathi and Ding. Signal distortion, power spectral density, correlation and autocorrelation.
- Wednesday : Begin Chapter 4. Analog modulation schemes.
- Lab next Friday : Finding and decoding NBFM