Lecture 3: Review of Signals and Systems, Time Domain

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Signals and Systems, Time Domain

Today:

- Types of signals
- Types of systems
- Measuring and characterizing signals
- Some important special signals
- A simple communication system signal: Pulse Dopper Radar
- Fourier series

Next Time:

- Frequency domain description of signals and systems

Signals and Systems, Part 1

- A *signal* is a real (or complex) valued function of one or more real variables.
  - voltage across a resistor or current through inductor
  - pressure at a point in the ocean
  - amount of rain at 37.4225 N, 122.1653 W
  - amount of rain at 16:00 UTC as function of latitude,longitude
  - price of Google stock at end of each trading day

  In this course the independent variable is almost always time.

- Physical signals have units, e.g., volts or psi (Si pascal $=$ N$/$m^2)

- Signals can (usually or in principle) be measured:
  - $g(t) \mapsto g(0)$ (value at specific time, 0)
  - $g(t) \mapsto \int_{-\infty}^{\infty} g(u)\, du$ (total area)
  - $g(t) \mapsto \int_{-\infty}^{\infty} |g(u)|^2\, du$ (total energy)

Signals and Systems (cont.)

- A *system* is an object that takes signals as inputs and produces signals as outputs.

$$g(t) \longrightarrow \boxed{\text{system}} \longrightarrow f(t)$$

- In general, the output signal depends on entirety of input signal; e.g.,
  - $f(t) = \frac{d}{dt} g(t)$
  - $f(t) = \int_{t-1}^{t} g(u)\, du$
  - $G(f) = \int_{-\infty}^{\infty} g(t) e^{-i2\pi f t}\, dt$
- Examples of physical systems:
  - Electrical circuit: voltage in, voltage or current out
  - Building: earth shaking in, building shaking out
  - Audio amplifier

Signal Energy and Power

- The *energy* of a signal $g(t)$ is

$$\int_{-\infty}^{\infty} |g(t)|^2 \, dt$$

  We are interested in energy only when it is finite. Common cases:

  - Bounded signal of finite duration; e.g., a pulse
  - Exponentially decaying signals (output of some linear systems with pulse input)

- Necessary conditions for finite energy.

  - The energy in the "tails" of the signal must approach 0:

$$\lim_{T\to\infty} \left(\int_{-\infty}^{T} |g(t)|^2] dt + \int_{T}^{\infty} |g(t)|^2] dt \right) = 0$$

Signal Energy and Power

- The power of a signal is defined as a limit:

$$P_g = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |g(t)|^2 \, dt$$

  This is the average energy per unit time.

- This limit may be 0 even when the signal is not. For example

$$g_1(t) = \begin{cases} e^{-at} & t > 0 \\ 0 & t < 0 \end{cases} \quad \text{or} \quad g_2(t) = \frac{1}{1+|t|}$$

- These both have a finite energy, so the power goes to zero as T goes to infinity.
- If $g(t)$ is complex valued, then $|g(t)|^2$ is the square of magnitude/modulus, $g(t)g^*(t)$.

Signal Energy and Power

- A signal is *periodic* if it repeats: $g(t+T) = g(t)$ for every t.
  E.g., $\sin t$ has period 2π and $\tan t$ has period π.
- The *power* of a periodic signal $g(t)$ is

  $$P_g = \frac{1}{T} \int_a^{a+T} |g(t)|^2 \, dt$$

  where T is the period of $g(t)$.
- Since all periods are the same, we can integrate our any period and get the same result.

Units of Power

- If a signal $g(t)$ measured in volts is applied to a load resistor R, then the power in watts is
$$P = \frac{g(t)^2}{R}$$
Normally we do not care about the load, so we normalize to $R = 1$.
- In many applications, the effect of the signal varies as the log of the signal; e.g., human hearing and sight.
- Power can be expressed in *decibels (dB)*, which are logarithmic and relative to some standard power. If P is measured in watts, then
  - power in dBW is $10 \log_{10} P$ (power relative to $1\,\mathrm{W}$)
  - power in dBm (or dBmW) is $\log_{10}(1000\,P) = 30 + 10 \log_{10} P$
- There are lots of other examples (dBA for accoustics, dBi for antennas)
- One bel (B) is too large to be useful.

The bel is named for Alexander Graham Bell (1847–1922). The dB was adopted by NBS in 1931. It is *not* an SI unit.

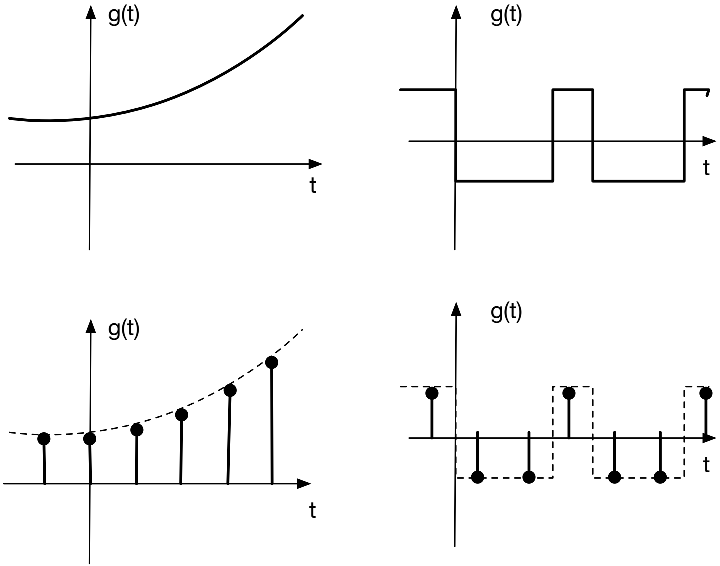
dB in Communications

- Expressing power in dB is very useful for communication systems
- Communication systems often have a wide range of amplitudes (1 kW transmitted, 1 μW received).
- Many of the components in the system have multiplicative effects (path loss, antenna gain)
- Example
  - Transmit 10 W ($+40$ dBm)
  - Transmit antenna gain of 10 ($+10$ dB)
  - Path loss of 10^9 (-90 dB)
  - Receive antenna gain 10 ($+10$ dB)
  - Received signal is 40 dBm $+$ 10dB - 90dB $+$ 10dB $=$ -30 dBm
- This is $10^{-30/10} = 10^{-3}$ mW, or 1 μW

Classification of Signals

- Signals can have a variety of characteristics, including
  - values can be continuous or discrete
  - continuous or discrete time variable
  - deterministic or random
- For deterministic signals, we have four cases:
  - continuous time, continuous valued (mathematics)
  - continuous time, discrete valued
  - discrete time, continuous valued (digital signal processing)
  - discrete time, discrete valued (digital switching)
- Time can be restricted to a finite interval (e.g., periodic)

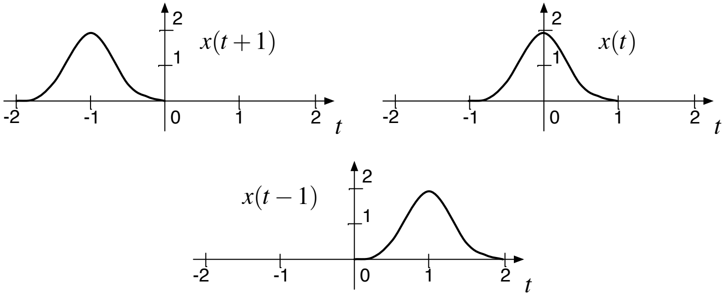
Classification of Signals (cont.)



Operations on Signals: Time Shift

For a continuous-time signal $x(t)$, and a time $t_1 > 0$,

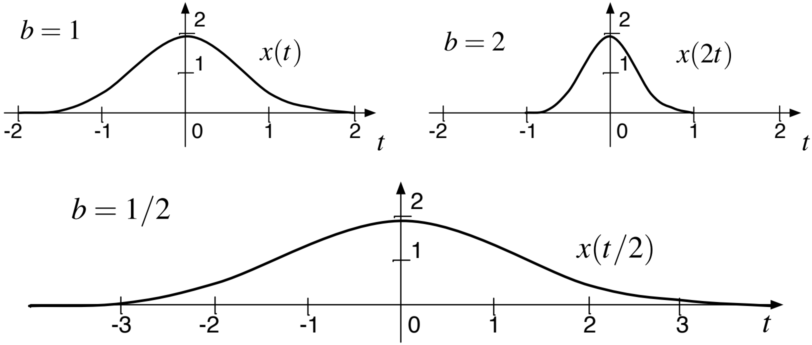
- Replacing t with $t - t_1$ gives a *delayed* signal $x(t - t_1)$
- Replacing t with $t + t_1$ gives an *advanced* signal $x(t + t_1)$



- May seem counterintuitive. Think about where $t - t_1$ is zero.

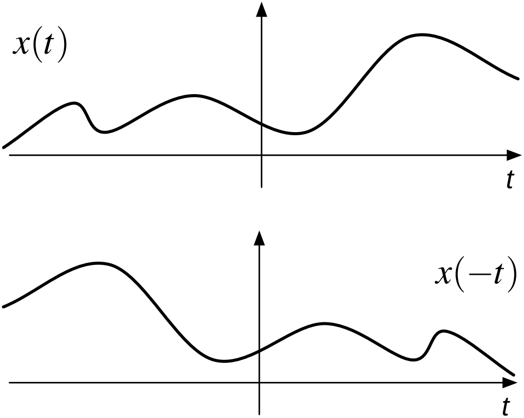
Operations on Signals: Time Scaling

A signal $x(t)$ is scaled in time by multiplying the time variable by a positive constant b, to produce $x(bt)$. A positive factor of b either expands $(0 < b < 1)$ or compresses $(b > 1)$ the signal in time.



Operations on Signals (cont.)

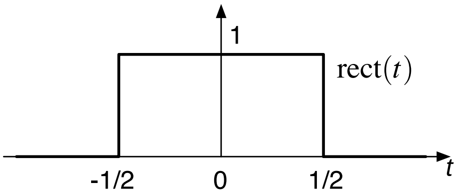
Replace t with $-t$, time reversed signal is $x(-t)$



Unit rectangle $\mathrm{rect}(t)$

Unit rectangle signal:

$$\mathrm{rect}(t) = \begin{cases} 1 & \text{if } |t| \leq 1/2 \\ 0 & \text{otherwise.} \end{cases}$$

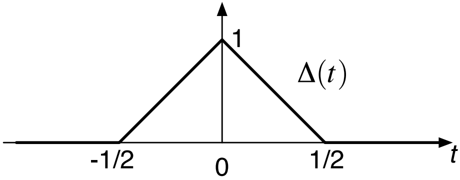


Also written as $\Pi(t)$.

Unit Triangle $\Delta(t)$

Unit triangle signal:

$$\text{rect}(t) = \begin{cases} |t| & \text{if } |t| \leq 1/2 \\ 0 & \text{otherwise.} \end{cases}$$



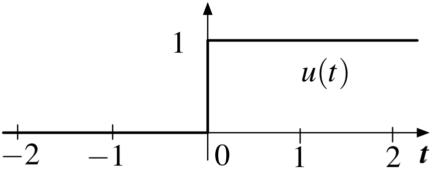
Also sometimes written as $\Lambda(t)$ or $\mathrm{tri}(t)$.

This is an unusual definition due to the book. Usually $\Delta(t)$ is 2 units wide, so that it is the convolution of two $\mathrm{rect}(t)$ functions.

Unit Step Function $u(t)$

- The Heaviside unit step function is defined by

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$



The unit step function corresponds to turning on at time 0.

- Unit step is integral of unit impulse:

$$u(t) = \int_{-\infty}^{t} \delta(u)\, du \;\Rightarrow\; \delta'(t) = u(t)$$

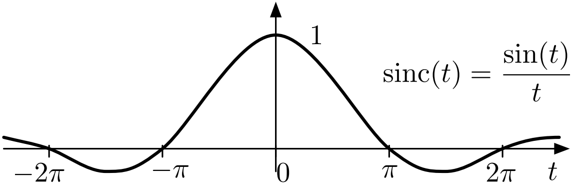
Oliver Heaviside (1850-1925) was a self-taught English electrical engineer, mathematician, and physicist who adapted complex numbers to the study of electrical circuits, invented mathematical techniques to the solution of differential equations (later found to be equivalent to Laplace transforms), reformulated Maxwell's field equations in terms of electric and magnetic forces and energy flux, and independently co-formulated vector analysis.

The sinc(.) function

- We will define the sinc(.) function differently than in EE102A, where we used the definition
$$\mathrm{sinc}_\pi(t) = \frac{\sin(\pi t)}{\pi t}$$
This is a function that is 1 a $t = 0$, and zero at the integers. We'll call this $\mathrm{sinc}_\pi(t)$ because it includes the π factor in its argument.
- In this course, we will define sinc(.) as
$$\mathrm{sinc}(t) = \frac{\sin(t)}{t}$$
This still has an amplitude of 1 at $t = 0$, but has zeros at multiples of π. The two are related by
$$\mathrm{sinc}(\pi t) = \mathrm{sinc}_\pi(t)$$

The sinc(.) function

This looks like this

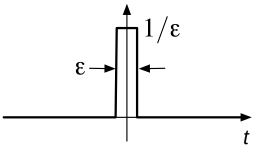


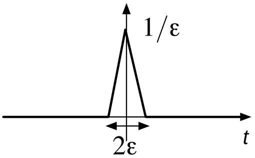
Unit Impulse Signal

(Dirac's) **delta function** or **unit impulse** δ is an *idealization* of a signal that

- is very large near $t = 0$
- is very small away from $t = 0$
- has integral 1

for example:

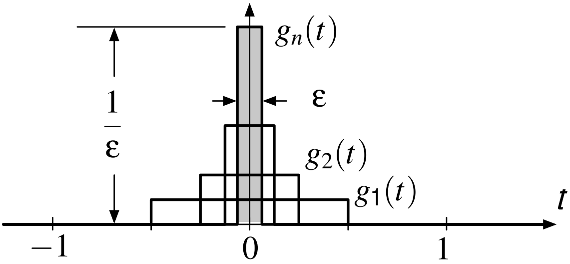




- the exact shape of the function doesn't matter
- ϵ is small (which depends on context)

Unit Impulse Signal

- Example: $g_n(t) = n\ \mathrm{rect}(nt)$ where n is an integer. As n gets larger



The area is one, but it gets narrower and narrower.

- Paul A. M. Dirac "defined" $\delta(t)$ by

$$\delta(t) \neq 0 \text{ if } t \neq 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

- The area of the impulse is important; the energy of $\delta(t)$ is not defined.

Properties of Unit Impulse Signal

- Sampling property:

$$\int_{-\infty}^{\infty} \varphi(t)\delta(t-T)\,dt = \int_{-\infty}^{\infty} \varphi(t+T)\delta(t)\,dt$$
$$= \int_{-\infty}^{\infty} \varphi(T)\delta(t)\,dt = \varphi(T)\int_{-\infty}^{\infty} \delta(t)\,dt = \varphi(T)$$

  In more rigorous mathematics, the sampling property *defines* the unit impulse as a *generalized function*.

- Convolution:

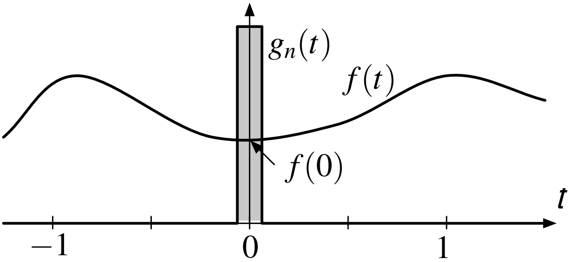
$$(\varphi * \delta)(T) = \int_{-\infty}^{\infty} \varphi(t)\delta(t-T)\,dt = \varphi(T)$$

Properties of Unit Impulse Signal

- Multiplication by a function:

$$\varphi(t)\delta(t) = \varphi(0)\delta(t)$$

- This is illustrated for some continuous function $f(t)$ as

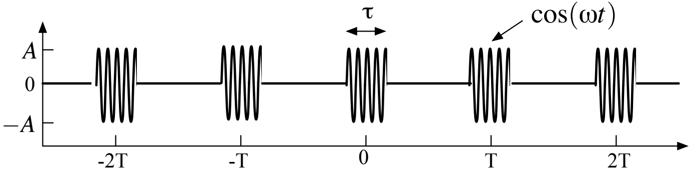


- If $f(t)$ is continuous, then the only value that matters is $f(0)$.

More Complex Signals

Many more interesting signals can be made up by combining simple signal elements.

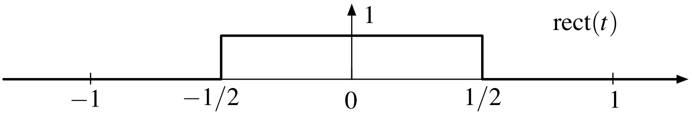
Example: Pulsed Doppler RF Waveform (we'll talk about this later!)



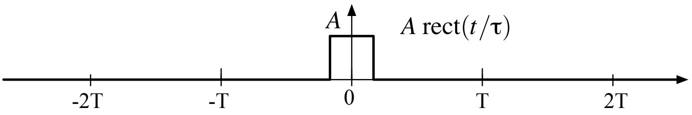
RF cosine gated on for τ μs, repeated every T μs, for a total of N pulses.

More Complex Signals

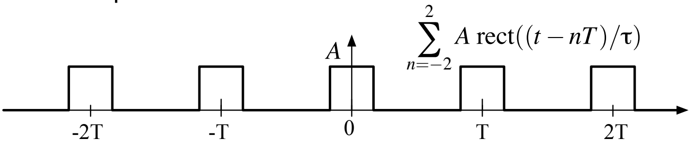
Start with a simple rect(t) pulse



Scale to the correct duration and amplitude for one subpulse



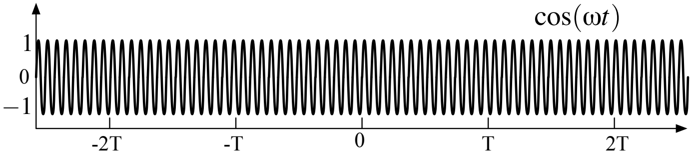
Combine shifted replicas



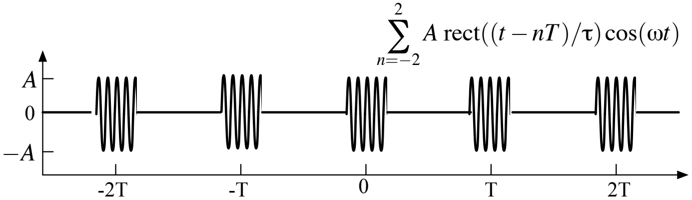
This is the *envelope* of the signal.

More Complex Signals

Then multiply by the RF carrier, shown below



to produce the pulsed Doppler waveform



Periodic Signals

- A signal g is called *periodic* if it repeats in time; i.e., for some $T > 0$,
$$g(t+T) = g(t)$$
for all t.
- If g is periodic, its period is the smallest such T.
- Examples: trignometric functions are periodic. Period of $\cos t$ is 2π; period of $\tan t$ is π.
- The period of $g(mt)$ is T/m.
- If g and f are periodic, their common period is $\mathrm{LCM}(T_g, T_f)$. E.g., period of $\sin \pi t + \sin 2\pi t/5$ is $\mathrm{LCM}(2, 5) = 10$.

Fourier Series

- Periodic signals can be written as the sum of sinusoids whose frequencies are integer multiples of the *fundamental frequency* $f_0 = 1/T_0$.
- The most general representation uses complex exponential functions.

$$g(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi f_0 n t}$$

  In general, the Fourier series coeffcients C_n are complex numbers, even when the signal is real valued. Note the factor of 2π since we are using the frequency in cycles/second, or Hz.
- The Fourier series coefficents can be computed by

$$C_n = \frac{1}{T_0} \int_a^{a+T_0} g(t) e^{-j2\pi f_0 n t} \, dt$$

  The integral is over any period of the signal.

Fourier Series Alternative Forms

- Euler's formula $e^j\theta = \cos\theta + j\sin\theta$ allows us to represent periodic signals as sums of sines and cosines:

$$g(t) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi}{T_0}nt\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi}{T_0}nt\right)$$

  The coefficients are

$$a_n = \frac{2}{T_0}\int_0^{T_0} g(t)\cos(2\pi f_0 nt)\,dt$$
$$b_n = \frac{2}{T_0}\int_0^{T_0} g(t)\sin(2\pi f_0 nt)\,dt$$

- A third compact form combines the sin(.) and cos(.) terms into phase shifted cos(.) terms

$$g(t) = C_0 + \sum_{n=1}^{\infty} A_n \cos(n2\pi f_0 t + \Theta_n)$$

  Each frequency component is described by amplitude and phase.

Fourier Series Examples

- Sinsuoids have a finite number of terms. By Euler's formula,

$$e^{it} = \cos t + i \sin t$$

Therefore

$$\cos t = \frac{e^{it} + e^{-it}}{2} \quad \text{and} \quad \sin t = \frac{e^{it} - e^{-it}}{2i}$$

The Fourier series coefficients for $\cos t$ are, $\ldots C_{-2}, C_{-1}, C_0, C_1, C_2, \ldots$

$$\ldots, 0, 0, \tfrac{1}{2}, 0, \tfrac{1}{2}, 0, 0, \ldots$$

and for $\sin t$ are

$$\ldots, 0, 0, -\tfrac{1}{2i} = \tfrac{i}{2}, 0, \tfrac{1}{2i} = -\tfrac{i}{2}, 0, 0, \ldots$$

Fourier Series of Square Wave

- Square wave with period 2π defined over interval $[-\pi, \pi]$ by

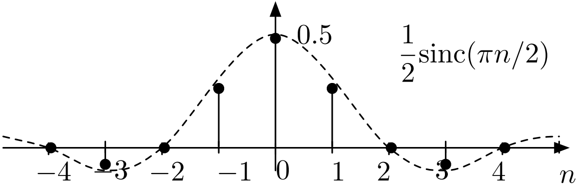
$$w(t) = \begin{cases} 1 & |t| < \pi/2 \\ 0 & -\pi < |t| < \pi/2 \end{cases}$$

Fourier series coefficients: if $n > 0$,

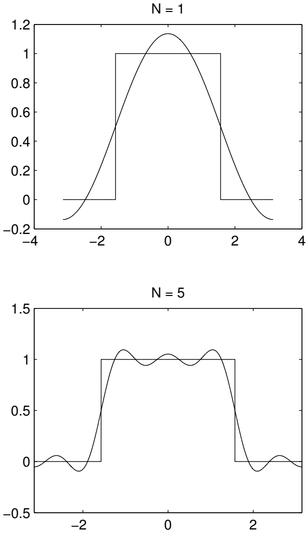
$$\begin{aligned} C_n &= \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} 1 \cdot e^{-jnt} \, dt = \frac{1}{2\pi} \frac{e^{-jnt}}{-jn} \Big|_{-\pi/2}^{\pi/2} \\ &= \frac{1}{2\pi} \frac{e^{\pi jn/2} - e^{-\pi jn/2}}{jn} \\ &= \frac{1}{2} \frac{\sin \pi n/2}{\pi n/2} = \frac{1}{2} \operatorname{sinc}(\pi n/2) = \frac{1}{\pi n} \quad (n \text{ odd}) \end{aligned}$$

Fourier Series of Square Wave

This looks like



Square Wave (cont.)



The overshoot is an example of the Gibbs' phenomenon.
The overshoot is $\approx 9\%$ and occurs no matter how many terms are used.

Next Time

- Fourier Transforms in $2\pi f$
- Some important theorems for communications
- Communications applications