

## Introduction to Communications

### EE 179: Problem Set # 7 - Winter 2007-08

Due at 5PM 13 March 2008

(20 points) 1. *Stereo FM*

When stereo FM was first introduced the FCC required that it be compatible with the older monaural FM, that is, the old radios had to still work. This problem considers the resulting system. Suppose that  $\ell(t)$  and  $r(t)$  are the left and right baseband audio channels, respectively. Each is assumed to be bandlimited to  $[-B, B]$ , where  $B = 15\text{kHz}$ .

Form a composite signal

$$m(t) = (\ell(t) + r(t)) + (\ell(t) - r(t)) \cos(2\pi f_{sc}t) + K \cos 2\pi f_p t$$

where  $f_p = 19\text{kHz}$  is the pilot carrier frequency and  $f_{sc} = 2f_p = 38\text{kHz}$ .

The signal  $m(t)$  is input to an FM modulator to form the transmitted signal  $x_c(t)$ . Give a labeled sketch of the spectrum  $M(f)$  of  $m(t)$  and identify the various components, including the monaural audio component.

At the receiver,  $x_c(t)$  is put into an FM discriminator. The output of the discriminator is in turn fed into three filters:

- a notch filter which extracts the pilot carrier. The output is put into a frequency doubler to produce the sinusoid for coherent demodulation of the output to the next filter.
- A band pass filter with center frequency  $f_{sc}$  and bandwidth 30 kHz (single-sided). The output is synchronously demodulated using the signal described above to produce an output of  $\frac{1}{2}(\ell(t) - r(t))$ .
- A low pass filter with bandwidth 15k Hz.

Describe how to get the separate monaural, left, and right signals from the above filter outputs.

(30 points) 2. *General Binary Detection* This problem considers a generalization of the derivation in class and the notes. It gives, for example, the error probability for ASK (in the usual unipolar case).

Given a binary random variable  $X$  with pmf

$$p_X(x) = \begin{cases} p & x = A_0 \\ 1 - p & x = A_1 \end{cases}$$

where  $p \in (0, 1)$  is a fixed parameter and  $A_0 < A_1$  are two real numbers (in class considered only  $A_1 = -A_0$ ). Let  $W$  be a Gaussian random variable, independent of  $X$ , with mean 0 and variance  $\sigma^2$ . Define  $Y = X + W$ .

- (a) Find and sketch  $f_{Y|X}(y|x)$  for all real  $y$  and  $x = A_0, A_1$ .

- (b) As in the purely discrete and purely continuous cases, there is a Bayes' rule for this mixed discrete/continuous case. In particular, we can define the conditional pmf for  $X$  given  $Y = y$  by

$$p_{X|Y}(x|y) = \frac{f_{Y|X}(y|x)p_X(x)}{f_Y(y)}$$

where  $f_Y(y) = \sum_x f_{Y|X}(y|x)p_X(x)$ .

Note that  $p_{X|Y}(x|y)$  is *not* an elementary conditional probability as emphasized in class because the conditioning event  $Y = y$  has zero probability.

One definition of an optimum detection rule for  $X$  given  $Y = y$  is the *maximum a posteriori (MAP)* detector, an example of a minimum average Bayes risk classifier in statistics. This is defined as the value of  $A_i$  which maximizes  $\Pr(X = A_i|Y = y) = p_{X|Y}(A_i|y)$  over  $i$ , that is, it chooses the most probable value of  $A_i$  given the observation  $Y = y$ . This is sometimes written

$$q(y) = \operatorname{argmax}_x p_{X|Y}(x|y)$$

Show that this is the same as the decision rule

$$\operatorname{argmax}_x f_{Y|X}(y|x)p_X(x)$$

that is, we do not have to compute  $f_Y(y)$  to use this rule.

Describe the MAP detector in as simple a form as possible for the case under consideration.

*Hint:* You are comparing to Gaussian densities with weights involving  $p$ . The comparison is unchanged by taking a logarithm, which simplifies the comparison rule. You should be able to show that your decision rule takes the form of a threshold rule, that is, decide that  $X = A_0$  if  $y < T$  and  $A_1$  otherwise, where you need to find  $T$ . Keep in mind that you can always get rid of things in the comparison that do not depend on which input was chosen, e.g., common  $y^2$  terms on both sides of the comparison.

Show that in the case  $p = 1/2$ , the following rule is optimal: Pick  $A_i$  if  $|y - A_i| < |y - A_j|$  (nearest neighbor or minimum distance classifier, pick the input closest to the observation). Also show in this case that the threshold  $T$  is midway between  $A_0$  and  $A_1$ .

- (c) Find an expression for the probability of error  $P_e$  using the MAP detector in terms of the  $Q$  function (general case, do not assume equal prior probabilities).

*Hint:* Begin with the general threshold  $T$  and only plug in your particular value when done.

Use this result to derive  $P_e$  for ASK (unipolar) using a detector that integrates and dumps the received signal and then does a threshold comparison with an optimal threshold.

(50 points) 3. *Bit error probability and SNR*

Let  $X_n$  be a random binary sequence where each symbol is equally likely to take on values of  $\pm 1$

$$m(t) = \sum_{k=-\infty}^{\infty} X_n p(t - nT)$$

where

$$p(t) = \begin{cases} 1 & 0 \leq t < T \\ 0 & \text{otherwise} \end{cases}$$

Let  $U$  be a uniform random variable, independent of the  $X_n$ , with pdf

$$f_U(u) = \begin{cases} \frac{1}{T} & 0 \leq u < T \\ 0 & \text{otherwise} \end{cases}$$

Define a random process

$$m(t) = \sum_{n=-\infty}^{\infty} X_n p(t - nT - U).$$

This process consists of a sequence of square pulses of length  $T$ , each taking on a random value of  $\pm 1$ . The starting time of the process is itself uniformly chosen in the first  $T$  seconds.

The process  $m(t)$  is modulated using double-sideband suppressed-carrier (DSB-SC) modulation to form a transmitted signal

$$X_c(t) = A_c m(t) \cos(2\pi f_c t),$$

It is assumed that  $f_c$  is a large integral multiple of  $1/T$ . This can be considered as PSK or as bipolar ASK.

This problem explores the properties of the baseband modulated binary sequence  $m(t)$  and a PSK modulated binary sequence  $X_c(t)$  and relates the probability of error in the digital modulation scheme to the overall reconstruction SNR when the binary sequence comes from simple quantization of an analog waveform.

- (a) Find the autocorrelation function  $R_m(t, s)$ . This is a classic problem in communications and is best done in several steps.
- i. Show that the autocorrelation function can be written as

$$R_m(t, s) = \sum_{n=-\infty}^{\infty} E(p(t - nT - U)p(s - nT - U)).$$

- ii. Show that  $R_m(t, s) = 0$  if  $|t - s| > T$ .
- iii. Suppose that  $|t - s| \leq T$  and assume for the moment that  $t > s > 0$ . The terms in the sum for the autocorrelation function all have the form

$$\frac{1}{T} \int_0^T p(t - nT - u)p(s - nT - u) du.$$

The integrand will be either 0 or 1, and it will be 1 if and only if  $u \in [0, T]$  is such that  $p(t - nT - u)p(s - nT - u) = 1$ . Show that this can only be the case if

$$t - (n + 1)T \leq u \leq s - nT$$

and that given  $t$  and  $s$  meeting the above conditions, this inequality can hold for only one possible value of  $n$ .

- iv. Use the results of the previous parts to find  $R_m(t, s)$ . You should have been able to show that it depends on  $t$  and  $s$  only through the difference  $t - s$ .

- (b) Find the power spectral density  $S_m(f)$ , the Fourier transform of the autocorrelation.
- (c) Estimate the bandwidth of the binary information signal  $m(t)$  by finding the distance between the zeros closest to the origin of  $S_m(f)$ . For the rest of the problem assume that this is the message bandwidth and that the signal can be well approximated if only the portion in  $[-B, B]$  is available.

- (d) Suppose that the receiver receives  $R(t) = A_r X_c(t) + N(t)$ , where  $A_r$  is a positive constant representing the channel loss and  $N(t)$  is an additive white zero mean Gaussian noise process with power spectral density  $N_0/2$ . The signal  $R(t)$  is passed through an ideal bandpass filter with center frequency  $f_c$  and bandwidth  $2B$  and then demodulated using a coherent ideal product demodulator to produce a signal

$$Y(t) = Km(t) + n(t).$$

Find the constant  $K$  and describe the noise  $n(t)$ . Sketch the power spectral densities  $S_Y(f)$  and  $S_n(f)$  (label the sketches).

Find the power in the information and noise terms and the signal-to-noise ratio SNR which is the ratio of these two powers.

- (e) So far we have looked at this as an analog modulation problem, but now consider reconstructing the bits. Suppose that the receiver is also coherent in the sense of knowing  $U$ . Consider the interval  $[0, T]$  which in the coherent case will see  $Y(t) = Km(t) + n(t) = KX_0 + n(t)$  during  $t \in [0, T]$ . (A similar analysis will work for any other interval  $[(n-1)T, nT]$ .) The signal  $Y(t)$  is integrated from 0 to  $T$ , when the output of the integrator is sampled to form a decision function

$$Z_1 = \int_0^T Y(t) dt$$

and the estimate is now

$$\hat{X}_0 = \begin{cases} +1 & Z_1 \geq 0 \\ -1 & Z_1 < 0 \end{cases}$$

Find an expression for the bit error probability  $P_e = \Pr(\hat{X}_0 \neq X_0)$  in terms of the  $Q$  function. Express  $P_e$  in terms of SNR found earlier.

- (f) Suppose now that the binary process  $X(t)$  is not the actual information signal, but a quantized version of it. For example, suppose that the original information process  $U(t)$  is assumed to be a zero mean stationary random process which is bandlimited to  $[-B, B]$  and has a marginal pdf  $f_{U(t)}(x)$  that is uniform on  $[-1/2, 1/2]$  and symmetric about 0 (an even function). The process is sampled to form a sequence  $U(nT)$ , where  $T < 1/2B$ . Since this is sampling higher than the Nyquist rate,  $U(t)$  can be recovered from the samples. Instead of communicating the original signal, however, we are only allowed to send one bit for each sample  $U(nT)$ . The binary sequence is produced by quantizing the samples using a binary quantizer

$$q(u) = \begin{cases} +a & u \geq 0 \\ -a & u < 0 \end{cases}$$

The samples are *encoded* into the binary signal by setting  $X_n = \text{sign}(q(U_{nT}))$ . The receiver then reconstructs an estimate of the original process as

$$\hat{U}_{nT} = a\hat{X}_n.$$

Find an expression for the overall mean-squared error  $E[(U_0 - \hat{U}_0)^2]$  in terms of the bit error probability  $P_e$ . (The same value will hold for  $E[(U_{nT} - \hat{U}_{nT})^2]$  for all  $n$ .) What value of  $a$  minimizes the mean-squared error?