

Introduction to Communications

Problem Set #6

Penultimate Problem Set

Due: Thursday, 6 March at 5 PM.

1. (20 points) Text Problem 3.26 (use lower sideband SSB).
2. (20 points) Consider the phase-shift method of generating SSB with the modification that the oscillator produces $\frac{A_c}{2} \cos(\omega_c t + \Theta)$, where Θ is a random variable uniformly distributed on $[0, 2\pi)$ which is independent of the input signal. Assume that the input signal $m(t)$ is a weakly stationary random process with autocorrelation function $R_m(\tau)$ and power spectral density $S_m(f) = \mathcal{F}(R_m)$. This problem develops the autocorrelation and power spectrum of the SSB signal

$$m_c(t) = m(t) \frac{A_c}{2} \cos(\omega_c t + \Theta) - m_H(t) \frac{A_c}{2} \sin(\omega_c t + \Theta)$$

where $m_h(t)$ is the Hilbert transform of $m(t)$ ($-\tilde{m}(t)$ in the class supplementary notes on SSB), the output of the phase-shift filter

$$H_Q(f) = -\tilde{H}(f) = \begin{cases} -j & f > 0 \\ j & f < 0 \end{cases}$$

with $m(t)$ as the input.

- (a) Show that the crosscorrelation function $R_{mm_h}(\tau) = E(m(t)m_H(t + \tau))$ is given by

$$R_{mm_H}(\tau) = h_Q(\tau) * R_m(\tau),$$

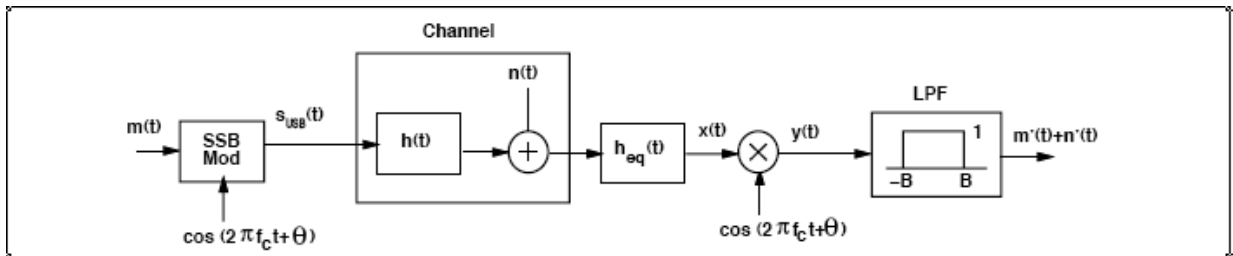
the convolution of the input autocorrelation and the filter h_Q . (This means that $R_{mm_H}(\tau)$ is the Hilbert transform of $R_m(\tau)$.)

- (b) Show that $R_{m_H}(\tau) = R_m(\tau)$, that is, the autocorrelation function of the Hilbert transform of the input signal is the same as that of the original input signal. (*Hint*: First consider the power spectral densities.)
- (c) Find an expression for $R_{m_c}(\tau)$ in terms of $R_m(\tau)$ and its Hilbert transform $\tilde{R}_m(\tau) = R_{mm_H}(\tau)$. Use this result to find an expression for the power spectral density $S_{m_c}(f)$ of the SSB signal. Sketch the result for the case where $S_m(f)$ is given by $1 - |f|/B$ for $|f| \leq B$ and 0 otherwise.

3. (20 points)

This problem involves a variation of the equalization problem from the midterm, but this time with SSB.

Consider the SSB system shown below, where the modulator is the USB version of an SSB modulator with output $s_{\text{USB}}(t) = .5A_c[m(t) \cos(2\pi f_c t) - m_h(t) \sin(2\pi f_c t)]$. The random phase offset θ in the carrier is known perfectly in the receiver, is uniform, and is independent of the noise $n(t)$.



Assume the noise $n(t)$ is white Gaussian noise with power spectral density $S_n(f) = N_0/2 = .1\text{mW/Hz}$. The PSD $S_m(f)$ of $m(t)$ is defined as

$$S_m(f) = \begin{cases} 10\text{mW/Hz} & |f| \leq .5B \\ 5\text{mW/Hz} & .5B < |f| \leq B \\ 0 & |f| > B \end{cases}$$

- What is the total power in $m(t)$?
- Find the SNR of the system output (power in $m'(t)$ over power in $n'(t)$) assuming $H(f) = H_{eq}(f) = 1$.
- Assume now that the frequency response of the channel $H(f)$ is given by

$$H(f) = \begin{cases} .1 & f_c - .5B \leq |f| \leq f_c + .5B \\ 1 & \text{else} \end{cases}$$

Find the equalizer $H_{eq}(f)$ such that in the absence of noise (i.e. for $n(t) = 0$), $m'(t) = m(t)$. Find the SNR with this equalizer design and also with $H_{eq}(f) = 1$.

- Explain qualitatively the tradeoffs involved in the design of the equalizer $H_{eq}(f)$.

4. (20 points) Text Problem 4.13.

5. (20 points) *FM Demodulation via Phase-lock loop*

The *phase-lock loop* (or *phase-locked loop* (PLL)) crops up in a variety of applications, including the FM demodulator considered here, the Costas loop used for coherent DSB-SC demodulation, and other applications in communications, controls, and signal processing. In this problem we derive the basic behavior using a linearized approximation.

The heart of a PLL is a *voltage-controlled oscillator* whose output (assuming an ideal device) is given by

$$r(t) = 2 \sin(2\pi f_c t + \psi(t))$$

where the rate of change of phase is proportional to the input:

$$\frac{d\psi(t)}{dt} = \alpha y(t).$$

In other words, the instantaneous frequency of the VCO output is a carrier frequency plus a term proportional to the input voltage. Thus a VCO can be viewed as an ideal frequency modulator.

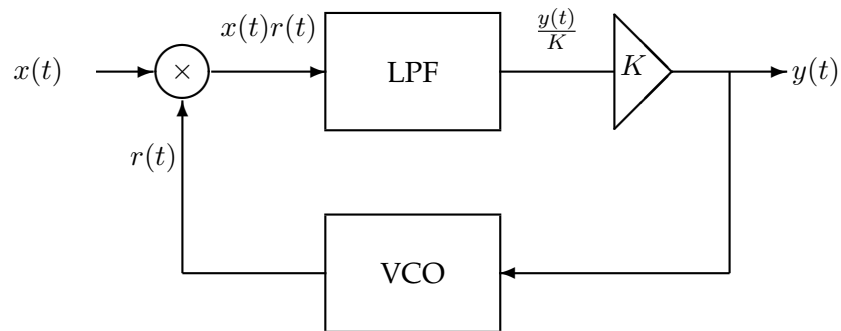


Figure 1: Phase-lock loop

Suppose that $x(t) = A \cos(2\pi f_c t + \phi(t))$. Suppose also that

- Both $\psi(t)$ and $\phi(t)$ are slowly varying with respect to $2\pi f_c t$.
- The lowpass filter (LPF) (also called the *loop filter*) is an ideal lowpass filter with a bandwidth wide enough to pass the low frequency components of $x(t)r(t)$ while rejecting terms for $|f|$ around $2f_c$.

(a) Show that $\psi(t)$ satisfies the nonlinear differential equation

$$\frac{d\psi(t)}{dt} = \alpha K A \sin(\psi(t) - \phi(t))$$

(b) The system is said to be in *lock* if the error $\psi(t) - \phi(t)$ is small, i.e.,

$$\psi(t) - \phi(t) \approx 0.$$

Show that when this is true, the system can be described in the frequency domain by

$$Y(f) = K_0 \frac{j2\pi f}{j2\pi f + \gamma} \Phi(f),$$

where you must derive K_0 and γ .

Discuss conditions on $d\phi(t)/dt$ or its transform under which $y(t)$ will be proportional to $d\phi(t)/dt$ so that indeed the PLL behaves as an ideal FM detector.