

Introduction to Communications

Problem Set #4

Due: Thursday, 14 February at 5 PM.

1. (30 points) *Probabilistic and time average autocorrelations of periodic random processes*

Suppose that $x_p(t)$ is a periodic signal with period T and time-average autocorrelation function

$$\mathcal{R}_{x_p}(\tau) = \frac{1}{T} \int_0^T x_p(t)x_p(t - \tau)dt$$

Suppose that U and A are independent random variables, where U has a uniform density on $[0, T]$ and A is Gaussian with 0 mean and variance σ^2 . Define a random process $X(t)$ by

$$X(t) = Ax_p(t - U)$$

that is, the random process looks like the deterministic periodic signal with a random delay and a random amplitude.

- (a) (15 points) Find the mean function $E[X(t)]$ and the autocorrelation function $R_X(t, s) = E[X(t)X(s)]$ of the random process $X(t)$. You should find a relationship between the time average autocorrelation function and the probabilistic autocorrelation function for such signals. Is the process weakly stationary?
- (b) (10 points) Specialize your answer to the special cases of $x_p(t) = \cos(2\pi t/T)$ and a square wave

$$x_p(t) = \frac{\cos(2\pi t/T)}{|\cos(2\pi t/T)|}$$

In particular, derive the form of the time-average and probabilistic average autocorrelations for both cases.

- (c) (5 points) Suppose that instead of the above description you are told that you have a weakly stationary random process $Y(t)$ with the property that its autocorrelation function $R_X(\tau)$ is periodic with period T , that is, $R_X(\tau + T) = R_X(\tau)$ for all τ . Does this necessarily imply that the random process itself is then periodic in the sense that $\Pr(|X(t) - X(t + T)| > \epsilon) = 0$ for any $\epsilon > 0$?

Hint: Use Chebychev's inequality.

2. (35 points) Suppose that $X(t)$ is a weakly stationary Gaussian random process with mean function $E[X(t)] = 0$ for all t and autocorrelation function $R_X(\tau)$. For any fixed T define a new random process

$$Y_T(t) = X(t) - X(t - T)$$

for all t . Note that T is considered a parameter here, t denotes the time index.

- (a) (10 points) For a fixed time t , find the mean $E[Y_T(t)]$, variance $\sigma_{Y_T(t)}^2$, characteristic function $\phi_{Y_T(t)}(\nu)$, and pdf $f_{Y_T(t)}(y)$ for the random variable $Y_T(t)$.
- (b) (15 points) Find the autocorrelation function $R_{Y_T}(t, s)$. Is the process WSS? If yes, find the power spectral density $S_{Y_T}(f)$.
- (c) (5 points) Describe the impulse response of an LTI system that with input process $X(t)$ will produce an output process $Y_T(t)$
- (d) (5 points) Define a new random process

$$Z_T(t) = \frac{Y_T(t)}{T}$$

Describe how the previous parts of this problem change in this case and what happens in the limit as $T \rightarrow 0$.

3. (20 points) *Phase modulation of a Gaussian process*

If $Y(t)$ is a complex-valued random process, its autocorrelation function is defined by

$$R_Y(t, s) = E[Y(t)Y^*(s)]$$

Let $X(t)$ be the WSS Gaussian random process of the previous problem. Define *phase modulation* of $X(t)$ as the random process $Y(t)$ given by

$$Y(t) = e^{j(\Delta X(t) + \Theta)}$$

where Θ is a random variable, independent of all of the $X(t)$, with a uniform distribution on $[0, 2\pi]$ and Δ is a modulation parameter.

Find the mean function and autocorrelation function of $Y(t)$. Is $Y(t)$ WSS?

Note: This is a rare example of where a nice closed form answer can be found for the autocorrelation of an angle modulated process. Phase modulation and frequency modulation are nonlinear modulation techniques, but for a Gaussian process one can find a simple solution.

4. (15 points) Haykin and Moher, problem 8.50.