

## Introduction to Communications

### Problem Set #3

Due: Thursday, 7 February at 5 PM.

1. (18 points)

Alice and Bob each choose at random a number between zero and two. We assume a uniform probability law under which the probability of an event is proportional to its area. Consider the following events:

*A*: The magnitude of the difference of the two numbers is greater than  $1/3$ .

*B*: At least one of the numbers is greater than  $1/3$ .

*C*: The two numbers are equal.

*D*: Alice's number is greater than  $1/3$ .

Find the probabilities  $P(A)$ ,  $P(B)$ ,  $P(A \cap B)$ ,  $P(C)$ ,  $P(D)$ ,  $P(A \cap D)$ .

2. (20 points) Suppose that  $X$  is a random variable with exponential pdf

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Given the various definitions of a new random variable  $Y = g(X)$  with  $g$  defined in each part, find the pdf  $f_Y(y)$ . Be sure to include the domain of definition for the argument.

(a)  $g(r) = |r|^2$ ,

(b)  $g(r) = r^{1/2}$ ,

(c)  $g(r) = \ln |r|$ ,

(d)  $g(r) = ar + b$ , where  $a$  and  $b$  are fixed constants.

(e) Find the pmf for the random variable  $W(r) = 3$  if  $r \geq 2$  and  $W(r) = 1$  otherwise.

3. (15 points) Suppose that  $X$  is a random variable described by an exponential pdf

$$f_X(\alpha) = \lambda e^{-\lambda \alpha}; \alpha \geq 0.$$

( $\lambda > 0$ .) Define a function  $q$  which maps nonnegative real numbers into integers by  $q(x) =$  the largest integer less than or equal to  $x$ . In other words

$$q(x) = k \text{ if } k \leq x < k + 1, \quad k = 0, 1, \dots$$

(This function is often denoted by  $q(x) = \lfloor x \rfloor$ .) The function  $q$  is a form of quantizer, it rounds its input *downward* to the nearest integer below the input. Define the following two random variables: the quantizer output

$$Y = q(X)$$

and the quantizer error

$$\epsilon = X - q(X).$$

Note: By construction  $\epsilon$  can only take on values in  $[0, 1)$ .

- (a) Find the pmf  $p_Y(k)$  for  $Y$ .
  - (b) Derive the probability density function for  $\epsilon$ . (You may find the total probability formula useful here, e.g.,  $P(G) = \sum_i P(G \cap F_i)$ , where the  $\{F_i\}$  form a a partition.)
  - (c) Find the mean  $E\epsilon$  and the mean squared error (MSE)  $E(\epsilon^2)$  of the quantizer.
4. (20 points)  $X, Y$  and  $Z$  are iid Gaussian random variables with  $\mathcal{N}(1, 1)$  distributions.

Define the random variables:

$$\begin{aligned} V &= 2X + Y \\ W &= 3X - 2Z + 5. \end{aligned}$$

- (a) Find  $E[VW]$ .
- (b) Find the 2 parameters that completely specify the random variable  $V + W$ .
- (c) Find the characteristic function of  $V$  and  $W$ .
- (d) Find the optimal (smallest MSE) affine estimator  $\hat{V}(W)$  of  $V$ , given  $W$ . That is, find  $a$  and  $b$  such that  $\hat{V}(W) = aW + b$  yields the minimum

$$\text{MSE} = E[(V - \hat{V}(W))^2]$$

5. (12 points) Two random variables  $X$  and  $Y$  have uniform probability density functions on  $(0, 1)$  and they are independent. Find the probability density function  $f_W(w)$  for the random variable  $W = (X - Y)^2$  and find the mean of  $W$ ,  $E(W)$ .
6. (15 points) Suppose that  $X_1, X_2, \dots, X_n$  are mutually independent Gaussian random variables with means  $\mu_i$ , variances  $\sigma_i^2$  and characteristic functions  $\phi_i(\nu) = e^{\nu\mu_i - \nu^2\sigma_i^2}$ .

Find and put in as simple a form as possible the mean, variance, characteristic function, and pdf of the random variable

$$Y = \sum_{i=1}^n X_i.$$