EE179 Spring 2008–2009

Introduction to Communications

Problem Set #7 Solutions

1. (25 points).
   Answer

   (a) The output following the differentiator is $s_0(t) = 10[2\pi f_c + 2\pi k_f m(t)] sin[2\pi f_c t + \int_0^t m(t)d\tau]$. For the envelope $10[2\pi f_c + 2\pi k_f m(t)]$ to be picked out properly, there can be no phase change, i.e., the envelope must remain non-negative. Given that our modulating signal $m(t) = 10cos(2\pi f_m t)$ has a maximum negative value of -10, the condition becomes $2\pi f_c + 2\pi k_f(-10) \geq 0$. Therefore $k_f \leq f_c/10 = 10^7$.

   (b) Carsons rule for estimating the bandwidth of FM transmission states $B = 2\Delta f + 2f_m$. Using $\Delta f = k_f A_m = (10)(10) = 100$ and $f_m = 3$kHz, Carsons rule give us 6.2 kHz as our transmission bandwidth. $\beta = \Delta f/f_m = 0.03 \leq 1$, the modulated wave is therefore a NBFM wave.

   (c) The instantaneous frequency $f+i$ of the modulated wave s(t) is given by $fc + k_f m(t) = 100 \times 10^6 + 100\cos(2\pi f_m t)$. The maximum and minimum values of fi are $(10010^6 + 100)$ Hz and$(100 \times 10^6 - 100)$Hz, respectively.

   (d) For an envelope detector to work properly, the charging time constant $R_s C$ has to be short compared to the carrier period $1/f_c$. This gives us $Rs \leq 10\Omega$. Choosing R_s to be 0.1Ω or less would work. Secondly, the discharging time constant$R_l C$ has to be much greater than the carrier period T_c of 10^{-8} sec but much smaller than the message period of $\frac{1}{3000}$ sec. Choosing R_l to be 1000 Ω satisfies this criterion. If $f_c \approx f_m$, there does not exist a R_l that satisfies this criterion. Therefore, envelope detection would not work under those circumstances.

2. (20 points).
   Answer

$$s(t) = \pm A_c \cos(2\pi f_c t)$$

The signal after the PM (product modulator) and before the integrator is given as:

$$x(t) = \pm A_c \cos(2\pi f_c t)\cos(2\pi f_c t + \phi) = \pm\frac{A_c[\cos(\phi) + \cos(4\pi f_c t + \phi)]}{2}.$$

Then the signal at the integrator output is given as:

$$y(t) = \int_0^{T_b} x(t)dt = \pm 0.5 A_c[T_b\cos(\phi) + \frac{1}{4\pi f_c}(\sin(4\pi f_c t + \phi) - \sin(\phi))].$$

If the noise samples are denoted by $n_0(t)$, then we have:

$$r(nT_b) = \pm 0.5A_c[T_b\cos(\phi) + \frac{1}{4\pi f_c}(\sin(4\pi f_c t + \phi) - \sin(\phi))] + n_0(nT_b).$$

If $f_c >> 1$, then the second term inside the square brackets can be neglected and $r(nT_b)$ is simplified as:

$$r(nT_b) = \begin{cases} 0.5A_cT_b\cos(\phi) + n_0(nT_b) & \text{if 1 is sent} \\ -0.5A_cT_b\cos(\phi) + n_0(nT_b) & \text{if 0 is sent} \end{cases}.$$

By symmetry, the threshold $T = 0$. If the noise $n_0(t)$ comes after the integrator, its samples are Gaussian random variables i.e. $n_0(nT_b) \sim N(0, 0.5N_0)$. Given that zeros and ones are equally likely, we can write the expression for probability of error as:

$$\begin{aligned} P_e &= 0.5 \times Prob(n_0(nT_b) > 0.5A_cT_b\cos(\phi)) + 0.5 \times Prob(n_0(nT_b) < 0.5A_cT_b\cos(\phi)) \\ &= Q(\frac{0.5A_cT_b\cos(\phi)}{\sqrt{0.5N_0}}) \end{aligned}$$

If we use the definition of Eb as given in the class, $E_b = 0.5A_c^2T_b$, we have

$$P_e = Q(\sqrt{\frac{E_bT_b\cos^2(\phi)}{N_0}}).$$

If the noise comes before the integrator, we have $n_0(nT_b) \sim N(0, 0.25T_bN_0)$, as shown in chapter 10.4 of the textbook. In this case we can obtain P_e following similar steps

$$P_e = Q(\sqrt{\frac{2E_b\cos^2(\phi)}{N_0}})$$

3. (20 points).
Answer

(a) Demodulation by $\cos(2\pi f_c t)$ reduces the power of the noise to $1/4 + 1/4 = 1/2$ times of the orginal power. From chapter 10.2 and 10.4 of the textbook, the statistics of $N(nT_b)$ following the integrator can be obtained as: $N(nT_b) \sim N(0, \frac{N_0T_b}{2}\frac{1}{2})$, i.e., $N(nT_b) \sim N(0, 0.25N_0T_b)$.

(b) To find $r_1(nT_b)$ (The $r(nT_b)$ when 1 is sent) : Let x(t) be the input to the integrator.

$$x(t) = A_c\cos 2(2\pi f_c t) + A\cos 2(2\pi f_c t) = \frac{1}{2}(1 + cos(4\pi f_c t))(A_c + A).$$

We have:

$$r_1(nT_b) = \frac{1}{2}(A_c + A)T_b.$$

(Note that the assumption, $f_cT_b >> 1$, was used).

Similarly we can find $r_0(nT_b)$ (The $r(nT_b)$ when 0 is sent) to be:

$$r_0(nT_b) = \frac{1}{2}(A - A_c)T_b.$$

(c) $p(\text{detect "1"}|\text{ send "0"}) = p(r_0(nT_b)+n(nT_b) > 0) = p(-0.5A_cT_b+0.5AT_b+n(nT_b) > 0)$ $= p(n(nT_b) > 0.5A_cT_b - 0.5AT_b)$. Given $N(nT_b) \sim N(0, 0.25N_0T_b)$, we have

$$p(\text{detect "1"}|\text{send "0"}) = Q(\sqrt{\frac{(A-A_c)^2T_b}{N_0}}).$$

(d) Following similar steps as part c), we have

$$p(\text{detect "0"}|\text{send "1"}) = Q(\sqrt{\frac{(A+A_c)^2T_b}{N_0}}).$$

(e)

$$\begin{aligned} P_b &= 0.5p(\text{detect "0"}|\text{send "1"}) + 0.5p(\text{detect "1"}|\text{send "0"}) \\ &= 0.5[Q(\sqrt{\frac{(A+A_c)^2T_b}{N_0}}) + Q(\sqrt{\frac{(A-A_c)^2T_b}{N_0}})]. \end{aligned}$$

(f) In the first case, $P_b = 0.5 \times 10^{-3} + 0.5(2 \times 10^{-7}) \approx 0.5 \times 10^{-3}$. In the second case, $P_b = 10^{-5}$.

4. (20 points).
Answer

(a) To find Y_1 we perform the integration of the input signal $s(t) = A_c \cos(2\pi f_1 t)$ (as $n(t) = 0$) against the local oscillator $cos(2\pi f_1 t)$.

$$Y_1 = \int_0^{T_b} A_c \cos(2\pi f_1 t)\cos(2\pi f_1 t)dt \approx \frac{A_cT_b}{2},$$

when $f_c >> 1$.
To find Y_2 we perform the corresponding integration:

$$Y_2 = \int_0^{T_b} A_c \cos(2\pi f_1 t)\cos(2\pi f_2 t)dt = 0,$$

where we have invoked the condition $f_1 - f_2 = .5/T_b$
Repeating the process for $s(t) = Accos(2\pi f_2 t)$ yields $Y_1 = 0$ and $Y_2 = \frac{A_cT_b}{2}$.
The signal after the summing junction is given as:

$$Y = Y_1 - Y_2.$$

For $s(t) = A_c \cos(2\pi f_1 t)$, $Y = \frac{A_c}{2T_b}$, leading to bit decision 1, while for $s(t) = A_c \cos(2\pi f_2 t)$, $Y = -\frac{A_c}{2T_b}$, leading to bit decision 0. The decision threshold should be set appropriately halfway in between these two levels at 0.

(b) When $f_1 - f_2 = .25/T_b$, perform the correlation integrals for $s(t) = A_c \cos(2\pi f_1 t)$. Y_1 remains $\frac{A_c T_b}{2}$, while $Y_2 = \frac{A_c T_b}{\pi}$. For $s(t) = A_c \cos(2\pi f_2 t)$. $Y_1 = \frac{A_c T_b}{\pi}$ while $Y_2 = \frac{A_c T_b}{2}$. Thus the inputs into the decision device are $Y = \pm A_c T_b(\frac{1}{2} - \frac{1}{\pi})$. We thus have degraded decision levels and less noise immunity, but our optimal decision level remains at 0.

(c) We are given that $N = N_1 - N_2$, which is the noise process entering into the decision device, is distributed as $N(0, 0.5N_0T_b)$. A bit decision will be in error if the magnitude of this noise process is greater than $A_cT_b/2$ when 0 is sent or less than $A_cT_b/2$ when 1 is sent. Therefore we have:

$$\begin{aligned} p(\text{ “0” detected}|\text{ “1” sent}) &= p(N + Y < 0|\text{ “1” sent}) \\ &= p(N + A_cT_b/2 < 0|\text{ “1” sent}) \\ &= Q(\sqrt{\frac{A_c^2 T_b}{2N_0}}). \end{aligned}$$

Similarly we can obtain:

$$\begin{aligned} p(\text{ “1” detected}|\text{ “0” sent}) &= p(N + Y > 0|\text{ “0” sent}) \\ &= p(N - A_cT_b/2 > 0|\text{ “0” sent}) \\ &= Q(\sqrt{\frac{A_c^2 T_b}{2N_0}}). \end{aligned}$$

(d)

$$P_b = 0.5p(\text{ “1” detected}|\text{ “0” sent}) + 0.5p(\text{ “0” detected}|\text{ “1” sent}) = Q(\sqrt{\frac{A_c^2 T_b}{2N_0}}).$$

Since $E_b = A_c^2 T_b/2$, we have

$$P_b = Q(\sqrt{E_b} N_0).$$

When $E_b/N_0 = 10\text{dB}$, $P_b = 0.023$. When $E_b/N_0 = 10\text{dB}$, $P_b = 7.8 \times 10^{-4}$.

5. Extra Credit Problem 1 (15 points Extra Credit)

Answer

When $f_c >> 1$,

$$r(t) = \int_0^1 m(t)\cos^2(2\pi f_c t)dt + n(t) \approx \frac{m(t)}{2} + n(t).$$

Suppose “1” was transmitted. In this case, $m(t) = 1$, so $r(t) = 1/2 + n(t)$. The decision device will make a decision error if $Re[r(t)] \leq 0$. Since $n(t) = 1.1e^{j\theta}$, we have $Re[r(t)] = Re[1/2 + n(t)] = 1/2 + 1.1\cos(\theta)$. And

$$\begin{aligned} P(Re[r(t)] \leq 0|1 \text{ is sent}) &= Pr(1/2 + 1.1\cos(\theta)|1 \text{ is sent}) \\ &= Pr(\theta = \frac{2\pi}{3}, \pi, \text{or} \frac{4\pi}{3}|1 \text{ is sent}) \\ &= 0.5. \end{aligned}$$

Following similar steps, we have:

$$P(Re[r(t)] \geq 0|0 \text{ is sent}) = 0.5.$$

Therefore the probability of making a decision error in the device is:

$$P_e = 0.5P(Re[r(t)] \geq 0|0 \text{ is sent}) + P(Re[r(t)] \leq 0|1 \text{ is sent}) = 0.5 \times 0.5 + 0.5 \times 0.5 = 0.5.$$

6. Extra Credit Problem 2(15 points Extra Credit) Text, Problem 10.29.

   Answer

   From textbook page 419 and table 10.1, we have the error rate for BPSK signal as $P_b = Q(\sqrt{\frac{2E_b}{N_0}})$. If we use three-bit repetition coding, the error occurs when more than two bits are in error. We have:

   $$P_e = C_3^2 P_b^2(1 - P_b) + P_b^3 = 3Q^2(\sqrt{\frac{2E_b}{N_0}})(1 - Q(\sqrt{\frac{2E_b}{N_0}})) + Q(\sqrt{\frac{2E_b}{N_0}})^3.$$

   (Discussion: If joint decoding of all three received bits are available. We can sum the received signal and the amplitude of the received signal will be three times the original amplitute, corresponding to a nine times increase in the power of received signal, while the power of the noise is only increased to three times of the original noise power. Thus the error rate of such joint decoding scheme is:

   $$P_e = Q(\sqrt{\frac{2E_b \times 9}{3N_0}}) = Q(\sqrt{\frac{6E_b}{N_0}}).$$

   We can see that joint decoding yields lower probability of error than symbol-by-symbol decoding but may have higher complexity at the decoder.)