EE179 Spring 2008–2009

Introduction to Communications

Problem Set #4 Solutions

1. (20 points)

   (a) Since τ is uniformly distributed between $[0, T]$, $x(t)$ is equally likely to be -1 or 1, so $E[x(t)] = (-1)(\frac{1}{2}) + (1)(\frac{1}{2}) = 0$.

   (b) Since $x(t)$ is periodic with period T, $R(t_1, t_2)$ will be periodic with period T too. For $|t_2 - t_1| \leq \frac{T}{2}$,

$$\begin{aligned} R(t_1, t_2) &= E[x(t_1)x(t_2)] \\ &= (1)P\{x(t_1)x(t_2) = 1\} + (-1)P\{x(t_1)x(t_2) = -1\} \\ &= \frac{1}{T}\int_0^T x(t_1 + t)x(t_2 + t)dt \\ &= \frac{1}{T}\int_0^T x(t_1 + t)x(t_1 + (t_2 - t_1) + t)dt \\ &= 1 - \frac{4|t_2 - t_1|}{T}. \end{aligned}$$

   Extending it over the period of T we obtain the $R(t_1, t_2)$.

   (c) $E[x(t)] = 0$ is independent of t, and the autocorrelation $R(t_1, t_2)$ is a function of the time difference $= t_2 - t_1$, so $x(t)$ is WSS.

   (d) Since the process is WSS, the autocorrelation in a period can now be written as $R_x(\tau) = 1 - 4\frac{|\tau|}{T}$, $\tau \leq |\frac{T}{2}|$. The periodic $R_x(\tau)$ is plotted in Figure 1.

   (e) The PSD of $x(t)$ is the Fourier transform of its autocorrelation. The autocorrelation is periodic, so the PSD will be a sampled version of the transform of one period of $R_x(\tau)$. One period of $R_x(\tau)$ can be expressed as

$$2\text{tri}(\frac{2\tau}{T}) - \text{rect}(\frac{\tau}{T})$$

   , which has the transform

$$T\text{sinc}^2(\frac{Tf}{2}) - T\text{sinc}(Tf)$$

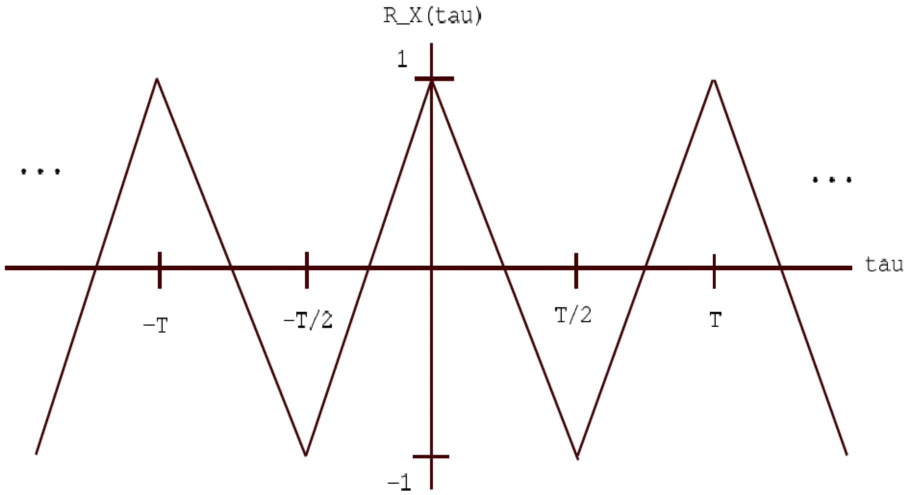

Figure 1: Autocorrelation of $x(t)$

so the PSD is

$$\begin{aligned} S_x(f) &= \mathcal{F}\{R_x(\tau)\} \\ &= \mathcal{F}\{(2\text{tri}(\frac{2\tau}{T}) - \text{rect}(\frac{\tau}{T})) * (\sum_{n=-\infty}^{\infty} \delta(t-nT))\} \\ &= (T\text{sinc}^2(\frac{Tf}{2}) - T\text{sinc}(Tf)) \cdot (\frac{1}{T}\sum_{n=-\infty}^{\infty} \delta(f-\frac{n}{T})) \\ &= \sum_{n=-\infty}^{\infty} [\text{sinc}^2(\frac{T}{2}\frac{n}{T}) - \text{sinc}(T\frac{n}{T})]\delta(f-\frac{n}{T}) \\ &= \sum_{n=-\infty}^{\infty} [\text{sinc}^2(\frac{n}{2}) - \text{sinc}(n)]\delta(f-\frac{n}{T}) \end{aligned}$$

(f) If the filter is not assumed to include the impulse at $f = \frac{1}{T}$, then the output PSD is 0, since $S_x(0) = 0$. If the filter is assumed to include the impulse at $f = \frac{1}{T}$, then the output PSD is

$$S_z(f) = \text{sinc}^2(0.5)[\delta(f-\frac{1}{T}) + \delta(f+\frac{1}{T})].$$

2. (a)

$$P_x = \int S_x(f)df = 2AB$$

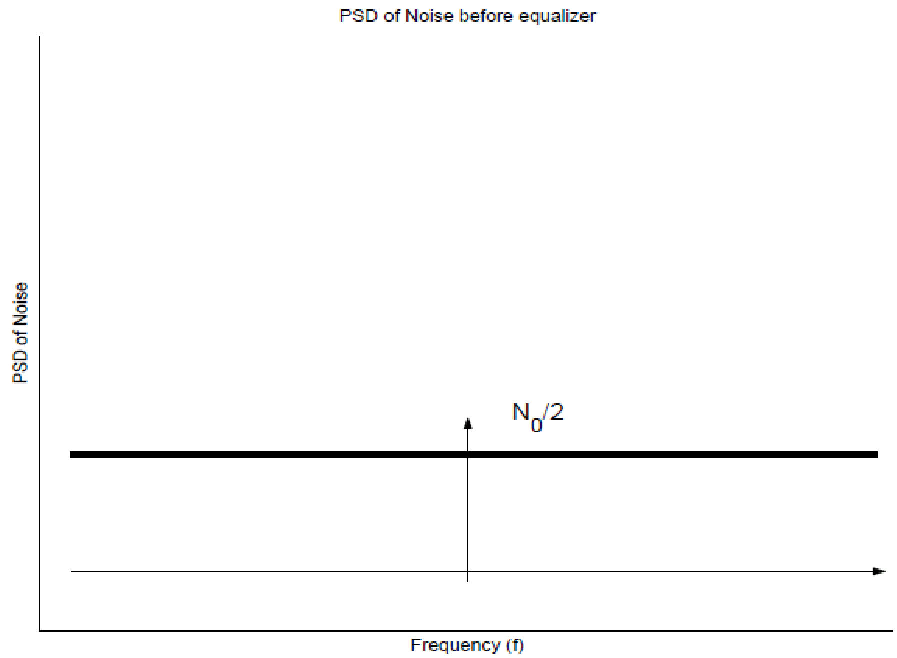

Figure 2: $S_N(f)$ before Equalizer

(b) $H_{eq}(f)$ should be designed so that $H(f)H_{eq}(f) = 1$ for $|f - f_c| \leq B$:

$$H_{eq}(f) = \begin{cases} 3 & |f - f_c| \leq \frac{B}{2} \\ 1 & \frac{B}{2} < |f - f_c| \leq B \\ 0 & \text{otherwise} \end{cases}$$

(c) The noise passes through the equalizer, demodulator and the LPF. The power spectral density of noise for these stages are shown in figures 2 through 5. Hence the Noise Power is $10N_0B$.

(d) Signal power as calculated from part a) is simply $2AB$. Noise power as calculated from part c) is $10N_0B$. Hence the SNR-ratio of Signal to Noise Power at the filter output is simply:

$$SNR = \frac{2AB}{10N_0B} = \frac{A}{5N_0}$$

(e) When the channel $h(t) = \delta(t)$, the signal power remains unchanged as the channel and equalizer still cancel the effect of each other and the multiplication by two cosines (modu- lation and demodulation) is balanced by the factor 2 in the LPF. The noise power, however, is changed to $2\frac{N_0}{2}2B = 2N_0B$. Thus the $SNR = \frac{2AB}{2N_0B} = \frac{A}{N_0}$.

3. (a) The mean of X is

$$E[X] = \int_0^T E[X(t)]dt + \int_0^{2T} .5E[X(t)]dt = 0$$

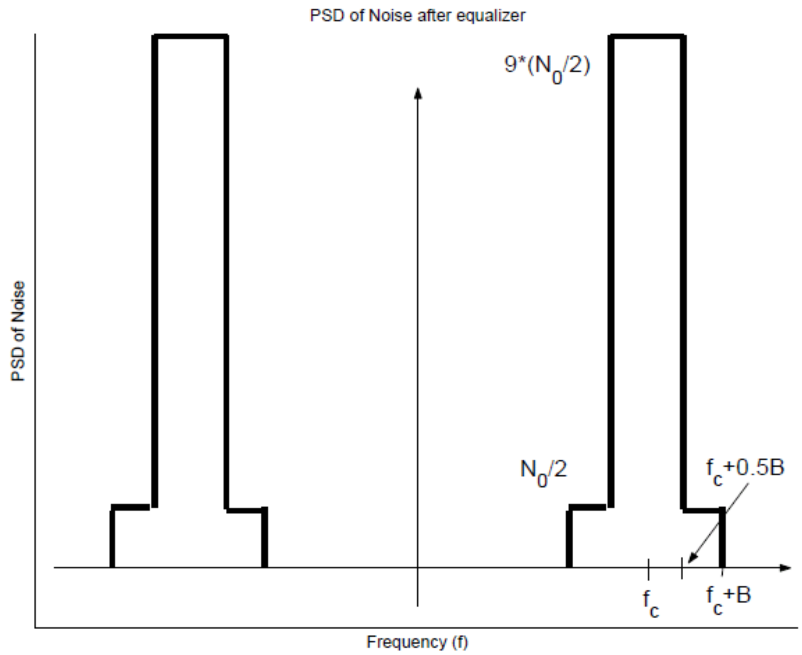

Figure 3: $S_N(f)$ after Equalizer

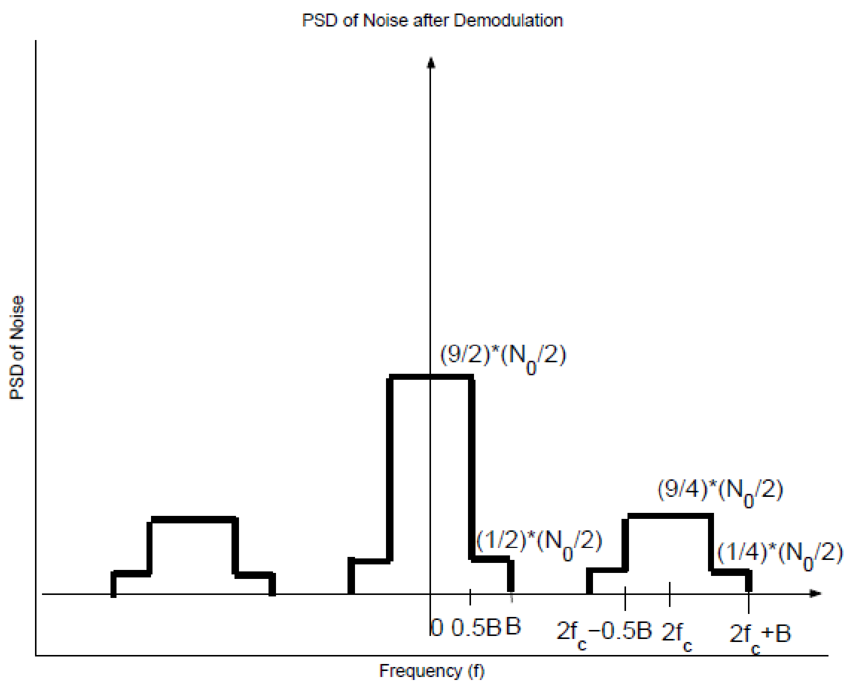

Figure 4: $S_N(f)$ after Demodulator

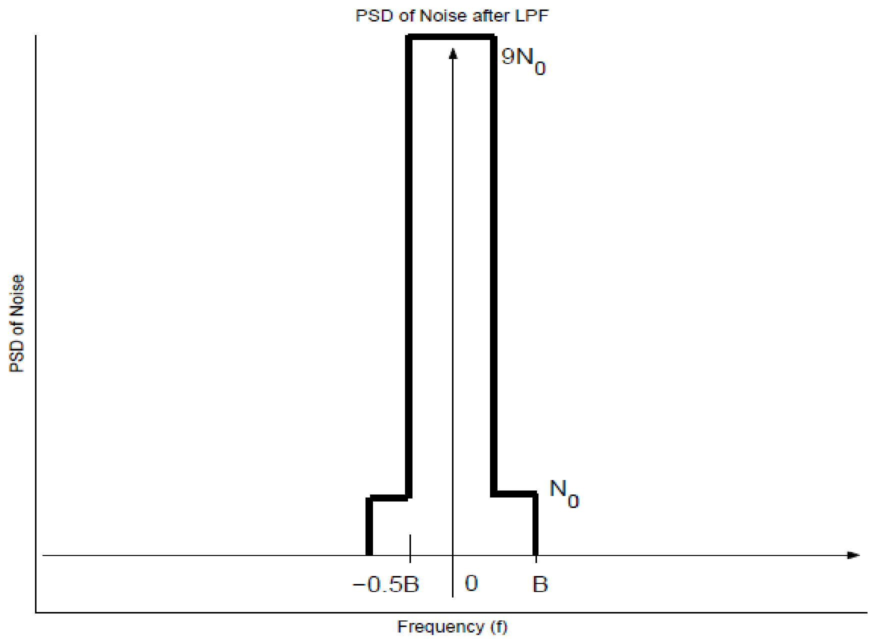

Figure 5: $S_N(f)$ after LPF

$$\begin{aligned}
\mathrm{Var}[X] &= E[X^2] - (E[X])^2 \\
&= E[X^2] \\
&= E[(\int_0^T X(t)dt + \int_0^{2T} .5X(t)dt)(\int_0^T X(u)du + \int_0^{2T} .5X(u)du) \\
&= \int_0^T \int_0^T E[X(t)X(u)]dtdu + \int_0^{2T} \int_0^T E[X(t)X(u)]dtdu + \int_0^{2T} \int_0^{2T} 0.25E[X(t)X(u)]dtdu
\end{aligned}$$

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Since $R_X(\tau) = 0.5N_0\delta(\tau)$, $E[X(t)X(u)] = 0.5N_0\delta(t-u)$. Substitute $E[X(t)X(u)]$ with $0.5N_0\delta(t-u)$ in the integral, we have

$$\mathrm{Var}[X] = \frac{5N_0T}{4}.$$

$$P(X \le 3) = F_X(3) = \frac{1}{2}[1 + \mathrm{erf}(\frac{x-\mu_X}{\sqrt{2}\sigma_X})] = \frac{1}{2}[1 + \mathrm{erf}(\frac{2}{\sqrt{\frac{5}{2}N_0T}})]$$

(b) The PSD of $Y(t)$ is

$$S_Y(f) = |H(f)|^2 S_X(f) = \frac{T^2N_0}{2}\mathrm{rect}(Tf),$$

and the autocorrelation is the inverse transform of $S_Y(f)$ is

$$R_Y(\tau) = \frac{TN_0}{2}\mathrm{sinc}(\frac{\tau}{T})$$

Since $S_Y(f)$ is not constant for all f, $Y(t)$ is not a white noise.

(c) The lowest τsuch that $R_Y(\tau) = 0$ is $\tau = T$. So samples of $Y(t)$ separated by T are uncorrelated, and since uncorrelated Gaussian random variables are also independent, samples of $Y(t)$ separated by T are also independent.

(d)

$$E[Y(3)Y(4.5)] = R_Y(4.5 - 3) = R_Y(1.5) = -\frac{1}{3\pi}.$$

4. (a) $R_X(\tau)$ is the inverse Fourier transform of $S_X(f) = \delta(f) + \text{tri}(\frac{f}{f_0})$:

$$R_X(\tau) = 1 + f_0 \text{sinc}^2(f_0\tau).$$

A sketch of $R_X(\tau)$ is shown in Figure 6.

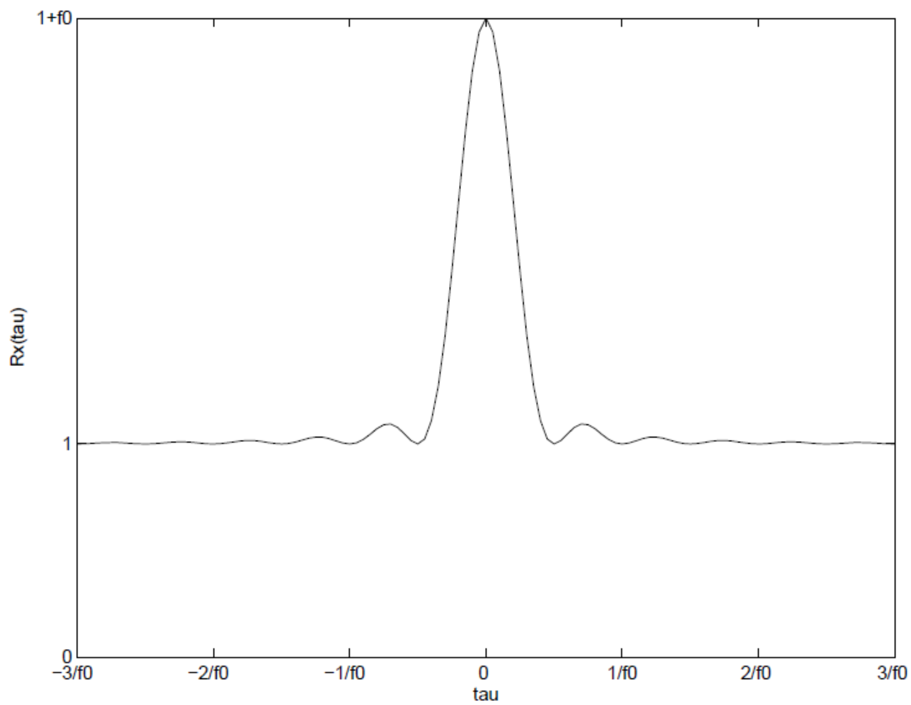

Figure 6: Autocorrelation of $X(t)$

(b) The dc power is given by

$$P_{dc} = \int_{0^-}^{0^+} S_X(f)df = 1.$$

(c) The ac power is given by

$$P_{ac} = R_X(0) - P_{dc} = 1 + f_0 - 1 = f_0$$

(d) Since $R_X(\tau)$¿ 0 for all τ, no sampling rate will give uncorrelated samples of X(t). Because independence implies uncorrelated samples, the fact that all samples are correlated means that none of the samples are independent.

5. Z is a linear functional of $X(t)$ and $Y(t)$, so Z is Gaussian. Thus we need to find the mean and variance in order to determine the distribution. By the linearity of expectation,

$$E[Z] = E[\int_0^2 (X(t) - 3Y(t) + 10)dt] = \int_0^2 (E[X(t)] - 3E[Y(t)] + 10)dt = 20$$

The mean square is given by

$$\begin{aligned} E[Z^2] &= E[(\int_0^2 (X(t) - 3Y(t) + 10)^2 dt)(\int_0^2 (X(u) - 3Y(u) + 10)^2 du)] \\ &= \int_0^2 (E[X^2] + E[Y^2] \\ &= \int_0^2 \int_0^2 E[X(t)X(u) - 3X(t)Y(u) + 10X(t) - 3X(u)Y(t) \\ &\quad + 9Y(t)Y(u) - 30Y(t) + 10X(u) - 30Y(u) + 100]dudt. \end{aligned}$$

Since $X(t)$ and $Y(t)$ are independent processes, E[X(t)Y (u)] = E[X(t)]E[Y (u)] = 0. Similarly, the other cross terms will evaluate to 0. So we have

$$\begin{aligned} E[Z^2] &= \int_0^2 \int_0^2 E[X(t)X(u)] + 9E[Y(t)Y(u)] + 100dudt \\ &= \int_0^2 \int_0^2 10\delta(t-u) + 9 \cdot 5\delta(t-u) + 100dudt \\ &= \int_0^2 \int_0^2 10 + 45 + 200dt \\ &= 510. \end{aligned}$$

So the variance $\text{Var}[Z] = E[Z^2] - (E[Z])^2 = 510 - 202 = 110$, and Z is a Gaussian random variable with mean 20 and variance 110.