EE179 Spring 2008–2009

Introduction to Communications

Problem Set #3 Solutions

1. (15 points) Text, Chapter 8, Problem 8.19 and 8.20.

   (1) 8.19
   Answer
   For two distinguishable dice (red and white), the number of outcomes is $6 \times 6 = 36$. For two nondistinguishable dice (red and white), the number of outcomes is $C_6^2 + C_6^1 = 21$.

   (2) 8.20
   Answer

   (a) There is one outcome with red = 5 and white = 2. Therefore the probability is $\frac{1}{\text{total number of outcomes}} = \frac{1}{36}$.

   (a) By symmetry of summation, the probablility of throwing a sum of 7 is the same for both distinguishable and non-distinguishable dices. We can solve either case and get the solution for the other case. It is easier to calculate the probability for distinguishable dices since in this case each outcome has the same probabilty of $\frac{1}{36}$ (For the non-distinguishable case different outcomes may have different probablities). For distinguishable dices, the number of outcomes that sums to 7 is 6. Therefore the probability is $\frac{1}{\text{total number of outcomes}} = \frac{6}{36} = \frac{1}{6}$

2. (15 points) Text, Chapter 8, Problem 8.22.
   Consider a random variable X defined by the double-exponential density

$$f_X(x) = ae^{-b|x|}, -\infty < x < \infty$$

   where a and b are constants.

   (a) Determine the relationship between a and b so that $f_X(x)$ is a probability density function.
   Answer
   A pdf should always integrate to 1 over all the possible values of the random variable, so let's integrate $f_X(x)$ from $-\infty$ to ∞:

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} ae^{-b|x|}dx \\
&= a\int_{-\infty}^{\infty} e^{-b|x|}dx \\
&= 2a\int_{0}^{\infty} e^{-bx}dx \\
&= -\frac{2a}{b}e^{-bx}\Big|_0^\infty \\
&= -\frac{2a}{b}(0-1) \\
&= \frac{2a}{b}
\end{aligned}
$$

In order for the pdf to integrate to 1, $2a = b$.

(b) Determine the corresponding distribution function $F_X(x)$.

Answer

To find the CDF, we integrate the pdf from its lowest possible value for the random variable. We need to consider 2 cases, of $x < 0$ and $x > 0$. Let's take the case of $x < 0$:

$$
\begin{aligned}
F_X(x) &= \int_{-\infty}^{x} ae^{-b|u|}du \\
&= a\int_{-\infty}^{x} e^{bu}du \\
&= \frac{a}{b}e^{bu}\Big|_{-\infty}^{x} \\
&= \frac{a}{b}(e^{bx}-0) \\
&= \frac{a}{b}e^{bx} \\
&= \frac{1}{2}e^{bx}
\end{aligned}
$$

For $x > 0$, we have:

$$
\begin{aligned}
F_X(x) &= \frac{1}{2} + \int_{0}^{x} ae^{-b|u|}du \\
&= \frac{1}{2} + \int_{0}^{x} ae^{-bu}du \\
&= \frac{1}{2} - \frac{a}{b}e^{-bu}\Big|_0^x \\
&= \frac{1}{2}(1-(e^{-bx}-1)) \\
&= \frac{1}{2}(2-e^{-bx}) \\
&= 1-\frac{1}{2}e^{-bx}
\end{aligned}
$$

(c) Find the probability that the random variable X lies between 1 and 2.
Answer
To find the probability that X lies between 1 and 2, we can just integrate the pdf between 1 and 2:

$$\begin{aligned}
f_X(x) &= \int_1^2 ae^{-b|x|}dx \\
&= a\int_1^2 e^{-bx}dx \\
&= -\frac{a}{b}e^{-bx}\Big|_1^2 \\
&= -\frac{1}{2}(e^{-2b} - e^{-b}) \\
&= -\frac{1}{2}e^{-b}(e^{-b} - 1)
\end{aligned}$$

3. (15 points) Text, Chapter 8, Problem 8.24.

(a) For $r \geq 0$,

$$\begin{aligned}
F_R(r) &= \int_0^r f_R(u)du \\
&= \int_0^r \frac{u}{b}exp(-\frac{u^2}{2b})du \\
&= \int_0^r -exp(-\frac{u^2}{2b})d(-\frac{u^2}{2b}) \\
&= \int_0^r -d(exp(-\frac{u^2}{2b})) \\
&= 1 - exp(-\frac{r^2}{2b}).
\end{aligned}$$

For $r \leq 0$, $F_R(r) = 0$. Therefore,

$$F_R(r) = \left\{ \begin{array}{ll} 1 - exp(-\frac{r^2}{2b}) & r \geq 0 \\ 0 & r \leq 1 \end{array} \right. .$$

(b)

$$\begin{aligned}
E[R] &= \int_0^\infty f_R(r) r dr \\
&= \int_0^\infty \frac{r^2}{b} exp(-\frac{r^2}{2b}) dr \\
&= \frac{1}{2}\int_{-\infty}^\infty \frac{r^2}{b} exp(-\frac{r^2}{2b}) dr \\
&= (\sqrt{\frac{\pi b}{2}}) \int_{-\infty}^\infty \frac{(\frac{r}{\sqrt{b}})^2}{\sqrt{2\pi}} exp(-\frac{(\frac{r}{\sqrt{b}})^2}{2}) d(\frac{r}{\sqrt{b}})
\end{aligned}$$

Let $y = \frac{r}{\sqrt{b}}$ and observe that $\int_{-\infty}^\infty \frac{y^2}{\sqrt{2\pi}} exp(-\frac{y^2}{2})dy$ is the mean square of a normal Gaussian random variable and equals 1, we have:

$$\begin{aligned}
E[R] &= (\sqrt{\frac{b\pi}{2}}) \int_{-\infty}^\infty \frac{y^2}{\sqrt{2\pi}} exp(-\frac{y^2}{2}) dy \\
&= \sqrt{\frac{b\pi}{2}}.
\end{aligned}$$

(c)

$$\begin{aligned}
E[R^2] &= \int_0^\infty f_R(r) r^2 dr \\
&= \int_0^\infty \frac{r^3}{b} exp(-\frac{r^2}{2b}) dr \\
&= -\int_0^\infty r^2 exp(-\frac{r^2}{2b}) d(-\frac{r^2}{b}) \\
&= -r^2 exp(-\frac{r^2}{2b})|_0^\infty + \int_0^\infty exp(-\frac{r^2}{2b}) d(r^2) \\
&= \int_0^\infty exp(-\frac{r^2}{2b}) d(r^2) \\
&= 2b
\end{aligned}$$

(d)

$$\begin{aligned}
\text{Var}[R] &= E[R^2] - (E[R])^2 \\
&= 2b - \frac{b\pi}{2}
\end{aligned}$$

4. (20 points)

(a) (4 points)
Answer

$$\begin{aligned} F_X(x) &= P(X \le x) \\ &= P(u^2 \le x) \\ &= P(u \le \sqrt{x}) \\ &= \int_0^{\sqrt{x}} 1 du \\ &= \sqrt{x} \end{aligned}$$

The pdf is just the derivative of this:

$$\begin{aligned} f_X(x) &= dF_X(x)/dx \\ &= d(\sqrt{x})/dx \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

The above are defined for $0 \le x \le 1$.

(b) (4 points)
Answer

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P(\sqrt{u} \le y) \\ &= P(u \le y^2) \\ &= \int_0^{y^2} 1 du \\ &= y^2 \end{aligned}$$

The pdf is just the derivative of this:

$$\begin{aligned} f_Y(y) &= dF_Y(y)/dy \\ &= d(y^2)/dy \\ &= 2y \end{aligned}$$

The above are defined for $0 \le y \le 1$.

(c) (4 points)
Answer

$$\begin{aligned} F_Z(z) &= P(Z \le z) \\ &= P(\ln u \le z) \\ &= P(u \le e^z) \\ &= \int_0^{e^z} 1 du \\ &= e^z \end{aligned}$$

The pdf is the derivative of $F_Z(z)$, which is again e^z.
The above are defined for $-\infty \le z \le 0$.

(d) (4 points)
Answer

$$\begin{aligned} F_V(v) &= P(V \leq v) \\ &= P(aZ + b \leq v) \\ &= P(a \ln u + b \leq v) \\ &= P(a \ln u \leq v - b) \\ &= P(\ln u \leq \frac{v-b}{a}) \\ &= P(u \leq e^{\frac{v-b}{a}}) \\ &= \int_0^{e^{\frac{v-b}{a}}} 1du \\ &= e^{\frac{v-b}{a}} \end{aligned}$$

The pdf is just the derivative of this:

$$\begin{aligned} f_V(v) &= dF_V(v)/dv \\ &= d(e^{\frac{v-b}{a}})/dv \\ &= \frac{1}{a} e^{\frac{v-b}{a}} \end{aligned}$$

The above are defined for $-\infty \leq v \leq b$.

(e) (4 points)
Answer
It's very easy to see that since u is uniformly distributed, W has a 50% chance of being either -1 or $+1$, and 0 chance of being anything else. Thus,

$$p_W(w) = \begin{cases} 0.5 & w = \pm 1 \\ 0 & otherwise \end{cases}$$

Writing this in terms of a discontinuous cdf, we get

$$F_W(w) = \begin{cases} 0 & w < -1 \\ 0.5 & -1 \leq w < 1 \\ 1 & w \geq 1 \end{cases}$$

5. (15 points) Text, Chapter 8, Problem 8.25.

(a) $F_X(x) = Pr\{X \leq x\} = Pr\{\sin(Z) \leq \ x\} = Pr\{Z \leq \arcsin(x)\}$. For $0 \leq x < 1$,

$$\begin{aligned} f_X(x) &= \frac{d}{dx}(F_X(x)) \\ &= f_Z(z)\frac{dx}{dz}|_{Z=\arcsin(x)} + f_Z(z)\frac{dx}{dz}|_{Z=\frac{\pi}{2}-\arcsin(x)}. \end{aligned}$$

Since $\frac{dx}{dz} = \frac{1}{\sqrt{1-x^2}}$,

$$\begin{aligned} f_X(x) &= f_Z(z)\frac{1}{\sqrt{1-x^2}}|_{Z=\arcsin(x)} + f_Z(z)\frac{1}{\sqrt{1-x^2}}|_{Z=\pi-\arcsin(x)} \\ &= \frac{1}{\pi\sqrt{1-x^2}}. \end{aligned}$$

Similarly for $-1 < x \le 0$,

$$\begin{aligned} f_X(x) &= \frac{d}{dx}(F_X(x)) \\ &= f_Z(z)\frac{dx}{dz}|_{Z=\pi+\arcsin(x)} + f_Z(z)\frac{dx}{dz}|_{Z=\frac{\pi}{2}-\arcsin(x)} \\ &= f_Z(z)\frac{1}{\sqrt{1-x^2}}|_{Z=\pi+\arcsin(x)} + f_Z(z)\frac{1}{\sqrt{1-x^2}}|_{Z=\frac{\pi}{2}-\arcsin(x)} \\ &= \frac{1}{\pi\sqrt{1-x^2}}. \end{aligned}$$

Therefore, $f_X(x) = \frac{1}{\pi\sqrt{1-x^2}}$ for $|x| < 1$.
By symmetry, $F_Y(y) = Pr\{Y \le y\} = Pr\{X \le y\} = F_X(y)$, therefore, $f_Y(y) = f_X(y) = \frac{1}{\pi\sqrt{1-y^2}}$ for $|y| < 1$.

(b) To show that X and Y are uncorrelated, we need to show that $\text{Cov}[X, Y] = 0$. It can be shown that:

$$\begin{aligned} \text{Cov}[X, Y] &= E(XY) - E(X)E(Y) \\ &= E(XY) - 0 \\ &= E(XY) \\ &= E(\sin(Z)\cos(Z)) \\ &= E(\frac{\sin(Z)\cos(Z)}{2}) \\ &= 0. \end{aligned}$$

(c) X and Y are not statistically independent. According to the definition of independence,X and Y are statistically independent if and only if $f_{X,Y}(x, y) = f_X(x)f_Y(y)$. To show that X and Y are not statistically independent, we only needs to give a counterexample where $f_{X,Y}(x, y) \neq f_X(x)f_Y(y)$.
Examine the neiborhood of the point $(x, y) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{10}})$, we have both $f_X(\frac{1}{\sqrt{2}}) > 0$ and $f_Y(\frac{1}{\sqrt{10}}) > 0$ but there is no $z \in [0, 2\pi]$ such that $\sin(z) = x$ and $\cos(z) = y$ and consequently $f_{X,Y}(x, y) = 0$. Since $f_{X,Y}(x, y) = 0$ and $f_X(x)f_Y(y) > 0$ we have found a counter example that $f_{X,Y}(x, y) \neq f_X(x)f_Y(y)$ near the neighborhood of $(x, y) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{10}})$. Therefore X and Y are not statistically independent.

6. (20 points)

(a) (5 points)
Answer

$$
\begin{aligned}
\Phi_{X_1}(\nu) &= \int_{-\infty}^{\infty} f_{X_1}(x) e^{j\nu x} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}} e^{j\nu x} dx \\
&= e^{j\nu\mu_1 - \frac{1}{2}\nu^2\sigma_1^2}
\end{aligned}
$$

In a similar way we can find $\Phi_{X_2}(\nu) = e^{j\nu\mu_2 - \frac{1}{2}\nu^2\sigma_2^2}$

(b) (5 points)
Answer
We note that the pdf of Y is given as the convolution of the pdfs of X1 and X2. Hence

$$f_Y(y) = f_{X_1}(x) * f_{X_2}(x).$$

We recall a property of the Fourier Transform that convolution in time-domain changes to multiplication in the frequency domain. Using this property and realizing that characteristic function is the Fourier Transform of the pdf, we get:

$$
\begin{aligned}
\Phi_Y(\nu) &= \Phi_{X_1}(\nu)\Phi_{X_2}(\nu) \\
&= e^{j\nu\mu_1 - \frac{1}{2}\nu^2\sigma_1^2} e^{j\nu\mu_2 - \frac{1}{2}\nu^2\sigma_2^2} \\
&= e^{j\nu(\mu_1+\mu_2) - \frac{1}{2}\nu^2(\sigma_1^2+\sigma_2^2)}
\end{aligned}
$$

(c) (5 points)
Answer
We can observe that $\Phi_Y(\nu)$ is same as the characteristic function of a Gaussian variable $\mathcal{N}(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$. Since the characteristic function is the inverse Fourier transform of the pdf and is unique, we have $Y \sim \mathcal{N}(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$, i.e., the distrubtino for Y is Gaussian with a mean of $\mu_1+\mu_2$ and a variance of $\sigma_1^2+\sigma_2^2$.

(d) (5 points)
Answer
Yes, we can generalize the result. Define:

$$Z = \sum_{i=1}^{N} X_i,$$

where $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$. In a similar way as part (b), we can find that

$$\Phi_Z(\nu) = e^{j\nu(\sum_{i=1}^{N}\mu_i) - \frac{1}{2}\nu^2(\sum_{i=1}^{N}\sigma_i^2)}.$$

Thus $Z \sim \mathcal{N}(\sum_{i=1}^{N}\mu_i, \sum_{i=1}^{N}\sigma_i^2)$. So we conclude that the sum of independent Gaussian random variables with any mean and any variance is again a Gaussian random variable with its mean equal to the sum of the means and variance equal to sum of the variances of the individual random variables. Note that this is a special property of the Gaussian distribution and is not true in general. For instance, is the sum of two Uniform random variables uniformily distributed ? The answer is no as we know that the pdf of the sum is a Triangle function and not the rect function.