EE179 Spring 2008–2009

Introduction to Communications

Problem Set #2 Solutions

1. (15 points) Bandwidth Efficient Modulation.
   Answer:
   (a) For Z(f), the spectrum of X(f) will be shifted to $\pm f_c$ and scaled by 0.5, as shown in Fig. 1.

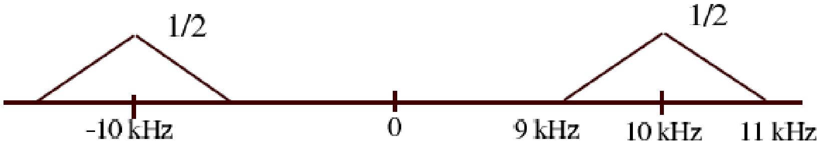

Figure 1: Z(f)

   (b) The filter will only pass the lower sidebands of Z(f), as shown in Fig. 2.

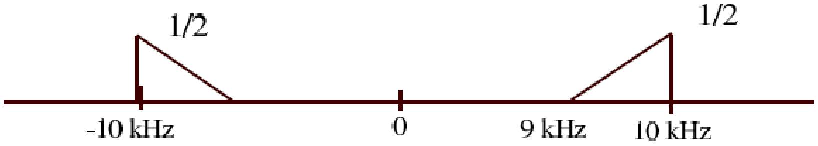

Figure 2: Y(f)

   (c) Setting $f_0 = f_c = 10$ kHz will produce a scaled version of X(f) along with higher frequency terms, as shown in the sketch of W(f) in Fig. 3. These terms can be removed by setting B = 1 kHz. Since the input has been multiplied by two cosines, it is scaled by .25, so G(f) should have a gain of A = 4.
   (d) Once again, setting $f_0 = f_c = 10$ kHz will produce a scaled version of X(f), along with higher frequency terms, as shown below in Fig. 4. Specifically,

$$x(t)\cos(2\pi f_c t)\cos(2\pi f_c t) \Leftrightarrow 0.25X(f-2f_c) + 0.5X(f) + 0.25X(f+2f_c).$$

   Setting B = 1 kHz will remove the terms centered around $\pm f_c$, while setting the gain of G(f) to be A = 2 will ensure that $\hat{x}(t) = x(t)$. Without H(f), the passband bandwidth of y(t) is 2 kHz. With H(f), the passband bandwidth of $y(t)$ is only 1 kHz.

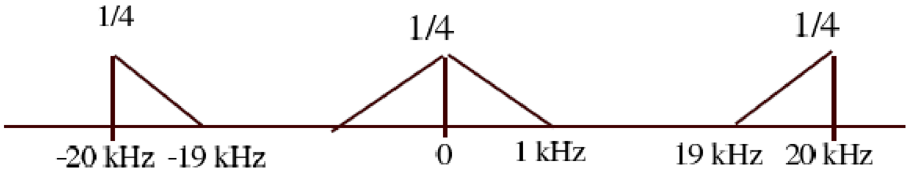

Figure 3: W(f)

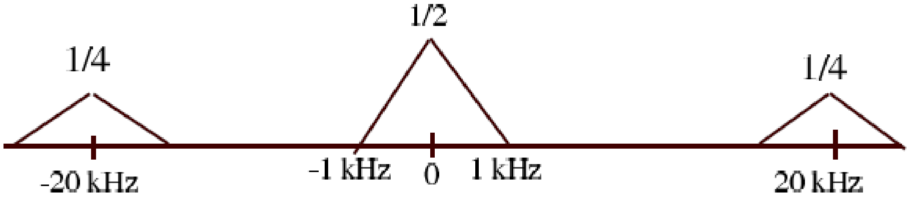

Figure 4: W(f) with H(f) removed

2. (15 points) Voice Scrambling.
   Answer:

   (a) The spectrum resulting from mixing x(t) with $\cos(2\pi ft)$ and high pass filtering is shown on the following page in Fig. 5.

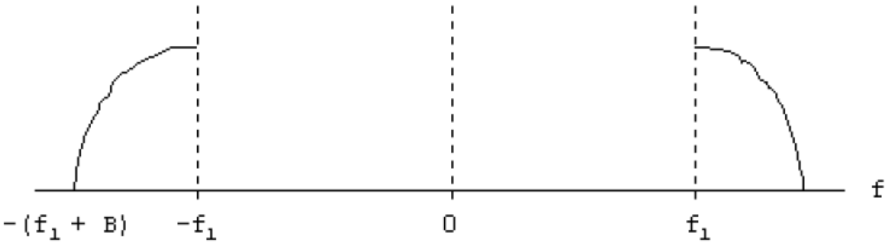

Figure 5: Y(f)

   (b) To obtain the spectrum Z(f), we need to shift the positive half of Y(f) down by $f_1 + B$ and the negative half of Y(f) up by $f_1 + B$. This can be achieved by setting the frequency f2 of the second mixer to be $f_1 + B$.

   (c) Theoretically, one could recover the original signal by passing the scrambled signal through the scrambler again. Another acceptable answer would be to multiply z(t) by another cosine of frequency B. Low-pass filtering and amplification by 2 would also yield the original spectrum X(f).

3. (15 points) Text, Chapter 2, Number 2.38
Answer:

   (a) By the definition of the autocorrelation function of power signals (real valued),

   $$R_g(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t-\tau)dt$$

   We notice that the non-DC coponents of $x(t)x(t-\tau)$, say $z(t)$, always returns a finite and bounded value from the integral $\int_{-T}^{T} z(t)dt$. Consequently their contribution to $R_g(\tau)$ will be zero when divided by $2T$, as $T \to \infty$. As a result,

   $$\begin{aligned} R_g(\tau) &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t-\tau)dt \\ &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [A_0^2 + \frac{A_1^2}{2}\cos(2\pi f_1 \tau) + \frac{A_2^2}{2}\cos(2\pi f_2 \tau)]dt \\ &= A_0^2 + \frac{A_1^2 \cos(2\pi f_1 \tau)}{2} + \frac{A_2^2 \cos(2\pi f_2 \tau)}{2} \end{aligned}$$

   (b) Plugging in $\tau = 0$ in $R_g(\tau)$ we get:

   $$Rg(0) = A_0^2 + \frac{A_1^2}{2} + \frac{A_2^2}{2}$$

   (c) We notice that the phase information conatined in θ_1 and θ_2 has disappeared.

4. (10 points) Text, Chapter 2, Number 2.41
Answer:

According to the definition of cross-correlation for real signals,

$$R_{12}(\tau) = g_1(\tau) * g_2(-\tau).$$

Decompose $g_2(t)$ to a linear combination of two rectangular functions such that $g_2(t) = -\operatorname{rect}(\frac{t}{6}) + 2\operatorname{rect}(\frac{t}{2})$.

$$\begin{aligned} R_{12}(\tau) &= g_1(\tau) * g_2(-\tau). \\ &= -\operatorname{rect}(\frac{\tau}{6}) * \operatorname{rect}(-\frac{\tau}{6}) + \operatorname{rect}(\frac{\tau}{6}) * 2\operatorname{rect}(-\frac{\tau}{2}) \\ &= -6\text{tri}(\tau/6) + [\operatorname{rect}(\frac{\tau}{2}) + \operatorname{rect}(\frac{\tau-2}{2}) + \operatorname{rect}(\frac{\tau+2}{2})] * 2\operatorname{rect}(-\frac{\tau}{2}) \\ &= -6\text{tri}(\tau/6) + 4\text{tri}(\tau/2) + 4\text{tri}((\tau-2)/2) + 4\text{tri}((\tau+2)/2 \end{aligned}$$

where $\text{tri}(t)$ is the standard triangular function. The sketch is shown in Fig. 6.

For real signal, $R_{21}(\tau) = g_2(\tau) * g_1(-\tau) = R_{12}(-\tau)$. Since $R_{12}(-\tau)$ is symmerical, $R_{21}(\tau) = R_{12}(\tau)$. These two signals are not orthogonal to each other since $R_{12}(0) \neq 0$.

5. (15 points) Text, Chapter 2, Number 2.47
Answer:

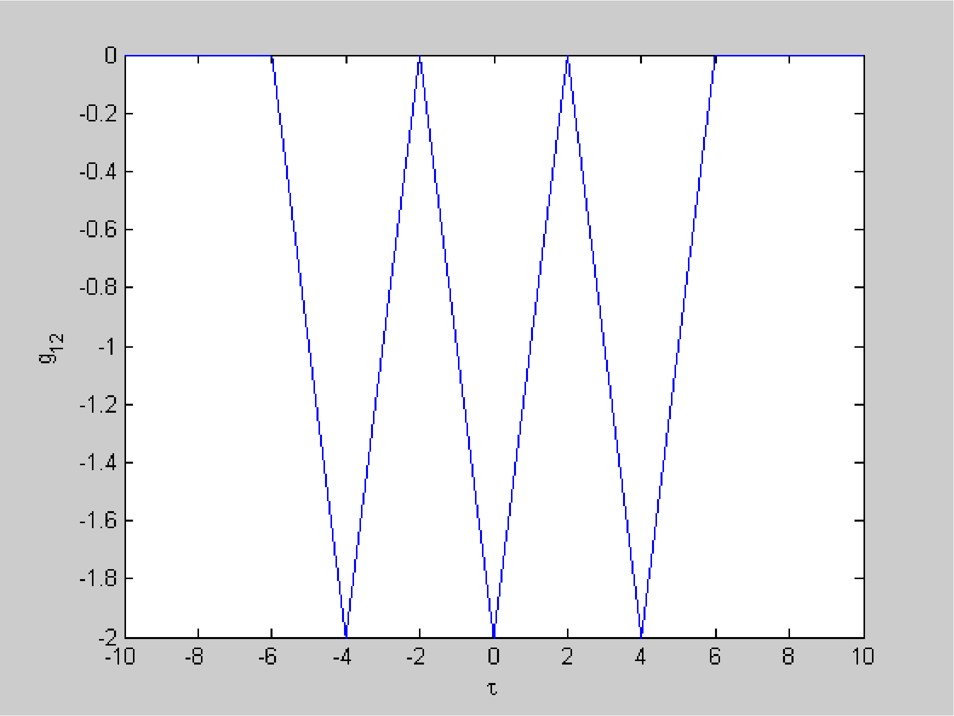

Figure 6: $g_{12}(\tau)$

(a)

$$\begin{aligned}
R_{g_p}(\tau) &= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} g_p(t)g_p^*(t-\tau)dt \\
&= f_c\int_{-\frac{1}{2f_c}}^{\frac{1}{2f_c}} A^2\cos(2\pi f_c t+\theta)\cos(2\pi f_c(t-\tau)+\theta)dt \\
&= f_c\int_{-\frac{1}{2f_c}}^{\frac{1}{2f_c}} \frac{A^2\cos(2\pi f_c\tau)}{2} + fraccos(4\pi f_c t+\theta) - 2\pi f_c\tau 2dt \\
&= f_c\frac{A^2\cos(2\pi f_c\tau)}{2f_c} + \frac{f_c}{4\pi f_c}(\cos(2\pi+\theta-2\pi f_c\tau) - \cos(-2\pi+\theta-2\pi f_c\tau)) \\
&= f_c\frac{A^2\cos(2\pi f_c\tau)}{2f_c} \\
&= \frac{A^2\cos(2\pi f_c\tau)}{2}
\end{aligned}$$

$R_{g_p}(\tau)$ is shown in Fig. 7 6.

(b) $R_{g_p}(0) = \frac{A^2\cos(2\pi f_c\times 0)}{2} = \frac{A^2}{2}$.

6. (15 points) Text, Chapter 2, Number 2.49
Answer:

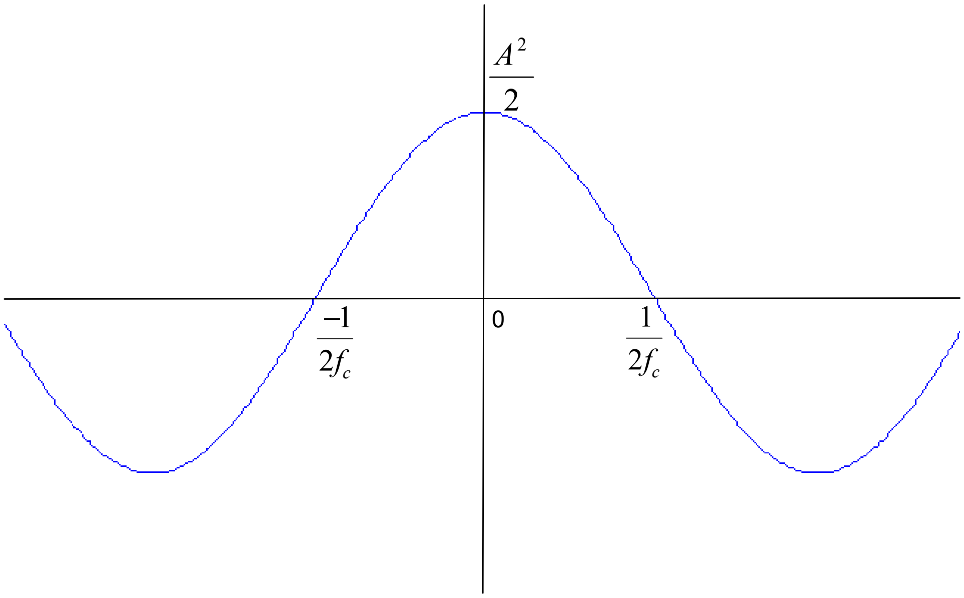

Figure 7: $g_{12}(\tau)$

(a) According to the Wiener-Khitchine Relation (Textbook Page 71), $R_{g_p}(\tau) \Leftrightarrow S_{g_p}(f)$. The power spectal density $S_{g_p}(f) = \mathcal{F}\{R_{g_p}(\tau)\} = \mathcal{F}\{\frac{A^2\cos(2\pi f_c\tau)}{2}\}$. Since $\mathcal{F}\{\frac{A^2\cos(2\pi f_c\tau)}{2}\} = \frac{A^2(\delta(f-f_c)+\delta(f+f_c))}{4}$, we have:

$$S_{g_p}(f) = \frac{A^2(\delta(f-f_c)+\delta(f+f_c))}{4}.$$

(b) For part(b),

$$\begin{aligned} R_{g_p}(\tau) &= \frac{1}{T_0}\int_{\frac{T_0}{2}}^{\frac{-T_0}{2}} g_p(t)g_p(t-\tau)dt \\ &= \frac{A_0^2}{2}\mathrm{tri}(\frac{2T}{T_0}). \end{aligned}$$

$$\begin{aligned} S_{g_p}(f) &= \mathcal{F}\{R_{g_p}(\tau)\} \\ &= \frac{A_0^2 T_0}{4}\,\mathrm{sinc}(\frac{T_0 f}{2}). \end{aligned}$$

7. (15 points)
Answer:

(a) Let $X_T(f)$ be the Fourier transform of the truncated signal $x_T(t)$. Using the frequency shifting properties of the cosine, we know that:

$$Y_T(f) = 0.25[X_T(f+f_1+f_2) + X_T(f+f_1-f_2) + X_T(f-f_1+f_2) + X_T(f-f_1-f_2)].$$

so that

$$|Y_T(f)|^2 = 0.0625|X_T(f+f_1+f_2)+X_T(f+f_1-f_2)+X_T(f-f_1+f_2)+X_T(f-f_1-f_2)|^2.$$

Applying the definition of the PSD, we get

$$\begin{aligned} S_y(f) &= \lim_{T\to\infty} \frac{1}{2T}|Y_T(f)|^2 \\ &= \lim_{T\to\infty} \frac{1}{2T} 0.0625|X_T(f+f_1+f_2)+X_T(f+f_1-f_2)+X_T(f-f_1+f_2)+X_T(f-f_1-f_2)|^2. \end{aligned}$$

(b) In order for this condition to be true, we need to make sure that there is no overlapping between the shifted versions of $X_T(f)$. Figure 8 gives a illustration of the power spectrum density of $S_y(f)$ when there is no overlapping. For the four bands to be non-overlapping, we need (1) $\delta f = f_1 - f_2 > B$, and (2) $f_2 > B$.

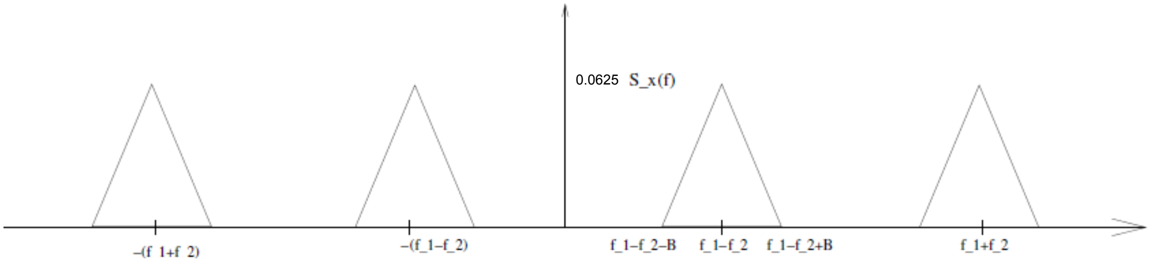

Figure 8: $S_y(f)$

(c) $A = \frac{1/2^2}{1/2^2} = \frac{1}{16}$ =0.0625.

(d) When $\delta f = 0$, $y(t) = x(t)\cos^2(2\pi f_1 t) = x(t)\frac{1+\cos^2(4\pi f_1 t)}{2}$. Assuming $f_1 > B$, we have:

$$\begin{aligned} S_y(f) &= \lim_{T\to\infty} \frac{1}{2T}|Y_T(f)|^2 \\ &= \lim_{T\to\infty} \frac{1}{2T}\{|\frac{X_T(f)}{2}|^2 + |\frac{X_T(f+2f_1)}{4}|^2 + |\frac{X_T(f-2f_1)}{4}|^2\}. \\ &= \frac{S_x(f)}{4} + \frac{S_x(f-2f_1)}{16} + \frac{S_x(f+2f_1)}{16}. \end{aligned}$$