Introduction to Communications

Problem Set #1 Solutions

1. LTI Filtering of Periodic Signals (10 points)
   Answer:

   We first have to find the Fourier series representation of the signal $x_p(t)$. We can do this using brute force integration, or by taking advantage of superposition. Note that the signal can be thought as the sum of a constant signal $x_1(t)$ with magnitude equal to -1, and a periodic signal $x_2(t)$ with the same period $T_0 = 8$, duration equal to 4 (from 0 to 4, 8 to 12, etc.), and amplitude equal to 2.

   For the first signal, it is obvious that the Fourier series representation is just -1 (so only $c_0^{(1)}$ is non-zero). For the second signal:

$$
\begin{aligned}
c_n^{(2)} &= \frac{1}{8}\int_{-4}^{4} x_2(t)exp(\frac{-j2\pi nt}{8})dt \\
&= \frac{1}{8}\int_{0}^{4} x_2(t)exp(\frac{-j2\pi nt}{8})dt \\
&= \frac{1}{8}\int_{0}^{4} x_2(t)exp(\frac{-j2\pi nt}{8})dt \\
&= \begin{cases} -\frac{2j}{\pi n} & \text{for n odd} \\ 0 & \text{for n even and } n \neq 0 \end{cases} \\
c_0^{(2)} &= 1
\end{aligned}
$$

   Fourier series representation of $x_p(t)$ is the sum of the Fourier series representation of $x_1(t)$ and $x_2(t)$:

$$
\begin{aligned}
c_n &= c_n^{(1)} + c_0^{(2)} \\
&= \begin{cases} -\frac{2j}{\pi n} & \text{for n odd} \\ 0 & \text{for n even} \end{cases}
\end{aligned}
$$

   consequently,

$$
x_p(t) = \sum_{k=-\infty}^{\infty} (-\frac{2j}{\pi(2k+1)})exp(\frac{j\pi(2k+1)t}{4})
$$

and

$$\begin{aligned}
y_p(t) &= \sum_{n=-\infty}^{\infty} H(nf_0)c_n exp(\frac{j2\pi nt}{T_0}) \\
&= \sum_{k=-\infty}^{\infty} \frac{\sin(8\pi\frac{2k+1}{8})}{2\pi\frac{2k+1}{8}}(-\frac{2j}{\pi(2k+1)})exp(\frac{j\pi(2k+1)t}{4}) \\
&= \sum_{k=-\infty}^{\infty} \frac{\sin((2k+1)\pi)}{2\pi\frac{2k+1}{8}}(-\frac{2j}{\pi(2k+1)})exp(\frac{j\pi(2k+1)t}{4}) \\
&= 0
\end{aligned}$$

A simple way to solve this problem is to identify that $H(nf_0) = 0$ for $n \neq 0$ and $c_0 = 0$. Therefore $y_p(t) = \sum_{n=-\infty}^{\infty} H(nf_0)c_n exp(\frac{j2\pi nt}{T_0}) = 0$.

2. Parseval's Relation (10 points)
   Answer:

   The Fourier transform of sinc(t) is rect(f). By parseval's theorem:

   $$\int_{-\infty}^{\infty} x^2(t)dt = \int_{-\infty}^{\infty} X^2(f)df$$

   substitute $x(t)$ with $sinc(t)$,

   $$\begin{aligned}
   \int_{-\infty}^{\infty} \text{sinc}^2(t)dt &= \int_{-\infty}^{\infty} \text{rect}^2(f)df \\
   &= \int_{-\frac{1}{2}}^{\frac{1}{2}} 1df \\
   &= 1
   \end{aligned}$$

3. Squared Sinusoids (10 points)
   Answer:

   $$\cos^2(2\pi f_c t) = \frac{1+\cos(4\pi f_c t)}{2}.$$

   Since $\cos(4\pi f_c t) \Leftrightarrow \frac{\delta(f-2f_c)+\delta(f-2f_c)}{2}$, $\mathcal{F}\{\cos^2(2\pi f_c t)\} = \frac{1}{2}\delta(f) + \frac{\delta(f-2f_c)+\delta(f-2f_c)}{4}$.

   For $\sin^2(2\pi f_c t)$, $\sin^2(2\pi f_c t) = \frac{1-\cos(4\pi f_c t)}{2}$. Similarly we have $\mathcal{F}\{\sin^2(2\pi f_c t)\} = \frac{1}{2}\delta(f) - \frac{\delta(f-2f_c)+\delta(f-2f_c)}{4}$.

4. Fourier Transform Properties (15 points)
   Answer:

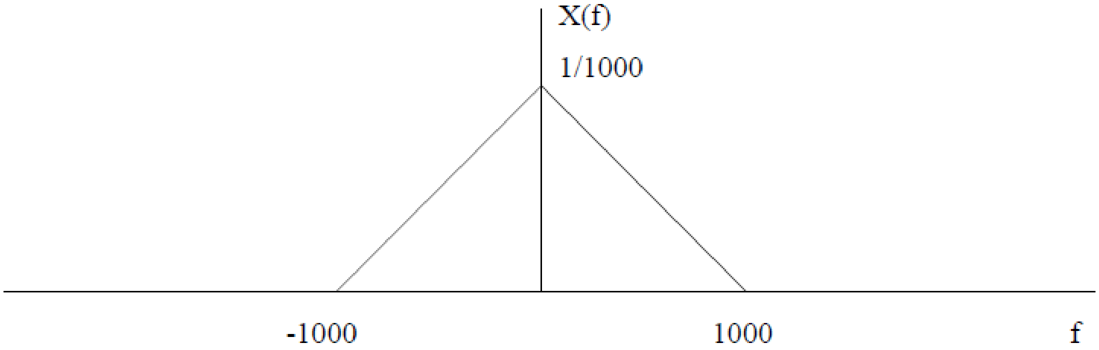

Figure 1: X(f)

(a) $g(t) = A\cos(\frac{\pi t}{T})\,\mathrm{rect}(t/T)$.

$$\begin{aligned}\mathcal{F}\{g(t)\} &= \frac{A}{2}(\delta(f-\frac{1}{2T})+\delta(f+\frac{1}{2T})) * T\,\mathrm{sinc}(Tf)\\ &= \frac{AT}{2}(\mathrm{sinc}(fT-1/2)+\mathrm{sinc}(fT+1/2)).\end{aligned}$$

For part (b) - (e), define: $g_0(t) = A\cos(\frac{\pi t}{T})\,\mathrm{rect}(t/T)$ and $G_0(f) = \frac{A}{2}(\mathrm{sinc}(fT-1/2)+\mathrm{sinc}(fT+1/2))$.

(b) $g(t) = g_0(t-\frac{T}{2})$. By the time-shifting property, $G(f) = \mathcal{F}\{g_0(t-\frac{T}{2})\} = e^{\frac{-j2\pi fT}{2}}G_0(f)$.

(c) for a half-sine pulse with a duration equal to aT,$g(t) = A\sin(\frac{\pi t}{aT})\,\mathrm{rect}(\frac{t-\frac{aT}{2})}{aT}) = g_0(\frac{t-\frac{aT}{2})}{a})$.

$$\mathcal{F}\{g(t)\} = ae^{-j\pi aTf}G(af).$$

(d) $g(t) = -g_0(t+\frac{T}{2})$. By the time-shifting and scaling property we have $G(f) = \mathcal{F}\{-g_0(t+\frac{T}{2})\} = -e^{\frac{j2\pi fT}{2}}G_0(f)$.

(e) $g(t)$ is the sum of the signal in part (b) and part(d). Therefore we can simply add up the two Fourier transforms in (b) and (d) together. $G(f) = e^{\frac{-j2\pi fT}{2}}G_0(f) - e^{\frac{j2\pi fT}{2}}G_0(f) = (e^{\frac{-j2\pi fT}{2}} - e^{\frac{j2\pi fT}{2}})G_0(f)$.

5. Modulation (20 points)
Answer:

(a) Since $\mathcal{F}\{\mathrm{sinc}^2(t) = tri(t)\}$, where $tri(t)$ is the standard triangular function, by the scaling law $X(f) = \mathcal{F}\{\mathrm{sinc}^2(1000t)\} = \frac{1}{1000}tri(\frac{f}{1000})$, shown in Fig. 1.

(b) $Z(f) = \mathcal{F}\{\mathrm{sinc}^2(1000t)\cos(2\pi f_c t)\} = X(f) * \frac{1}{2}(\delta(f-f_c)+\delta(f+f_c)) = \frac{1}{2}(X(f-f_c)+X(f+f_c))$, shown in Fig. 2

(c) $Y(f) = \frac{1}{2000}\,\mathrm{rect}(\frac{f}{1000})$ See Fig. 3

(d) $y(t) = \mathcal{F}^{-1}\{Y(f)\} = \mathcal{F}^{-1}\{\frac{1}{2000}\,\mathrm{rect}(\frac{f}{1000})\} = \frac{1}{2}\,\mathrm{sinc}(1000t)$,

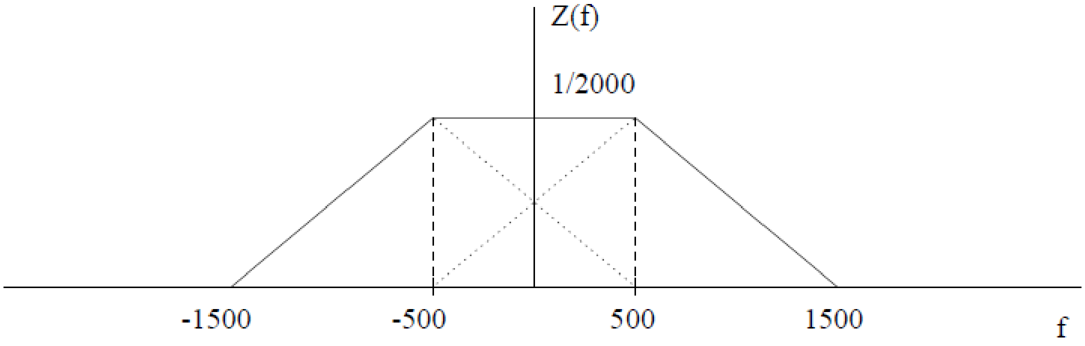

Figure 2: Z(f)

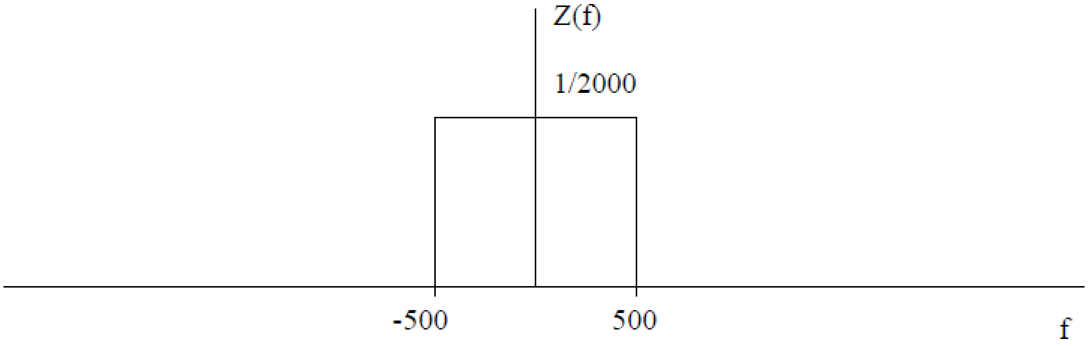

Figure 3: Y(f)

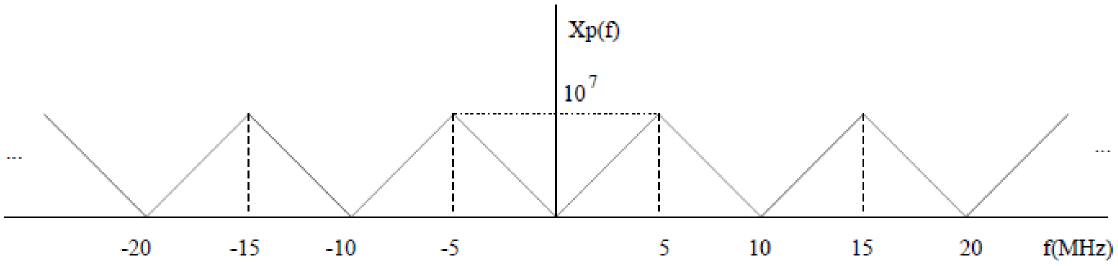

Figure 4: $X_p(f)$

6. Bandpass Sampling (15 points)
   Answer:

   (a) Since $x_p(t) = x(t)\delta_{T_s}(t)$, in the frequency domain $X_p(f) = X(f) * \mathcal{F}\{\delta_{T_s}(t)\} = X(f) * \frac{1}{T_s}\int_{n=-\infty}^{\infty}\delta(f - \frac{n}{T_s}) = \frac{1}{T_s}\int_{n=-\infty}^{\infty} X(f - \frac{n}{f_s})$, where $f_s = \frac{1}{T_s} = 10$ MHz. The schetch of $X_p(f)$ is shown in Fig. 4.

   Strictly speaking, there will be spikes of amplitude equal to 2×10^7 at frequencies ± 5 MHz, ± 15 MHz, ... However, since these spikes are countable and only occur at certain points (and not intervals) they dont have any effect on the inverse Fourier Transform. In any case, if you are concerned about those spikes you can define the filter transfer function to be equal to $\frac{1}{2}$ at the edges, which helps you get exactly the same spectrum as that of the original signal x(t).

   (b) From the spectrum of Fig. 4 we want to go back to the original spectrum. Therefore, we have to let pass only the frequency components in the intervals [-10 MHz,-5 MHz] and [10 MHz, 5 MHz]. Moreover, we have to scale by $T_s = 10^{-7}$ to compensate for the scaling that was introduced when we sampled the signal. Hence, $A = T_s = 10^{-7}, f_a = 5\mathrm{MHz}, and f_b = 10\mathrm{MHz}$.

7. Frequency Sampling (20 points)
   Answer:

   (a) See Fig. 5.

   (b) $x(t) = \mathcal{F}^{-1}\{X(f)\} = \sum_{n=-2}^{n=2} \exp(j2\pi n f_0 t) = 1 + 2\cos(2\pi f_0 t) + 2\cos(4\pi f_0 t)$, shown in Fig. 6.

   (c) Since $x(t) = \sum_{n=-2}^{n=2} \exp(j2\pi n f_0 t)$, $c_0 = c_{\pm1} = c\pm2 = 1$ and $c_n = 0$ elsewhere.

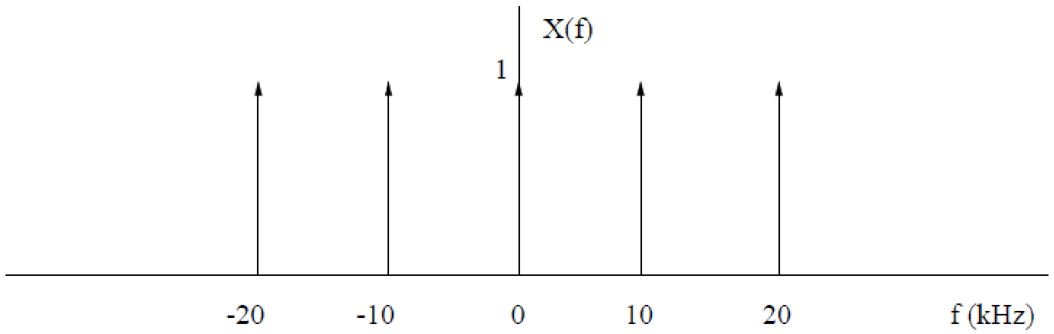

Figure 5: X(f)

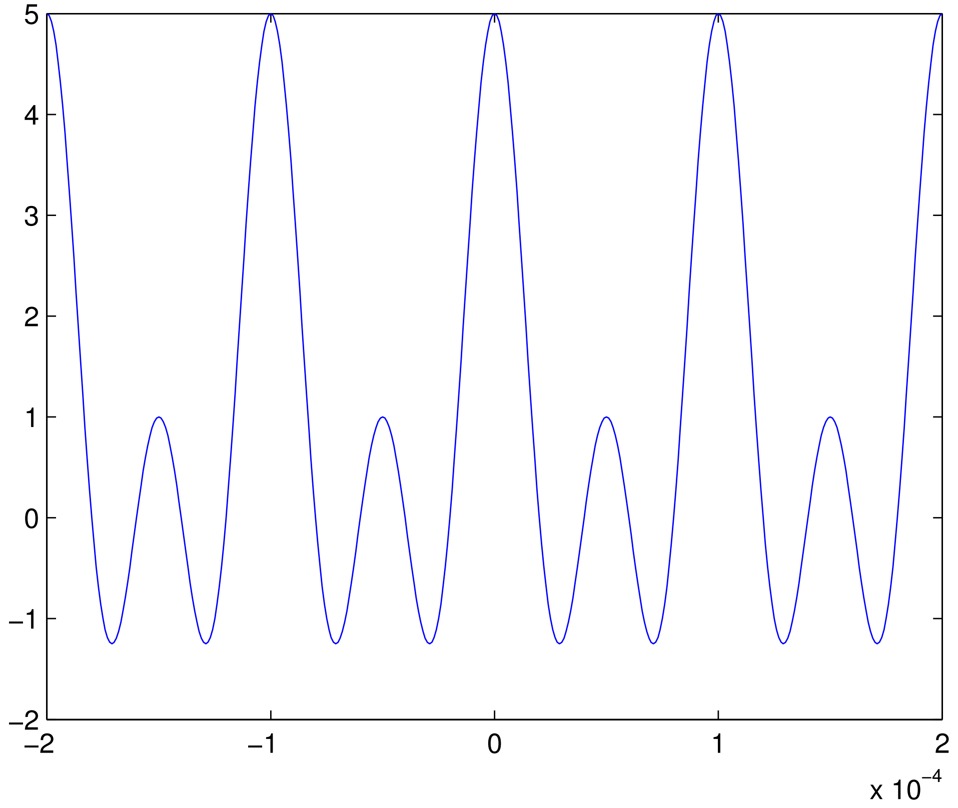

Figure 6: x(t)