Fourier Transforms

We have seen that it is possible to decompose a time or space signal into frequency components using Fourier series approaches. We can generalize much of this by considering a function, or transform, that can express any time or spatially dependent function into the frequency domain, where the amplitude of the signal is described as a function of frequency instead.

Although several such transforms exist, the most common one is the Fourier Transform, both because it is easily visualized as a frequency mapping and also because there exists a very efficient computer algorithm, the Fast Fourier Transform, or FFT, to compute it. For our purposes it will be sufficient to understand that Fourier transforms exist, memorize a few simple cases, and then implement it using fft code.

Definition

We'll use the following form of the Fourier transform, following Bracewell, because it has a very similar inverse form:

$$F(s) = \int_{-\infty}^{\infty} f(x)\, e^{-i 2\pi s x}\, dx$$

This is a one-dimensional form. $F(s)$ is the transform, that is the complex amplitude of the function at frequency s, $f(x)$ is the original function to be transformed, and $e^{-i2\pi sx}$ is the Fourier kernel. Recognizing that

$$e^{-i2\pi sx} = \cos 2\pi sx + i \sin 2\pi sx$$

and recalling our argument for determining Fourier series coefficients, we can interpret the transform $F(s)$ as representing the amplitude of the function $f(x)$ at frequency s. Having complex amplitudes and a complex kernel offers a compact way of dealing with both the sine and cosine parts.

Therefore, one way of thinking about the Fourier transform is as an analysis operation, where we break down the function $f(x)$ into its constitient parts.

We also have an inverse procedure for reassembling the Fourier components back into a recognizable function:

$$f(x) = \int_{-\infty}^{\infty} F(s)\, e^{+i2\pi s x}\, ds$$

This can be thought of as a synthesis operation, reconstructing a signal from its parts.

Two dimensions

The extension of the above to two dimensions requires that we have two orthogonal dimensions for frequency to correspond to the two orthogonal directions x and y in space. We usually denote these as u and v, and the units of each are cycles/meters if the x and y coordinates are meters. In 1-D, s is also cycles/m if x is meters.

The 2-D Fourier transform is

$$F(u,v) = \iint_{-\infty}^{\infty} f(x,y)\, e^{-i2\pi(ux+vy)}\, dx\, dy$$

Note again that the Fourier kernel can be expressed

$$e^{-i 2\pi (ux+vy)} = e^{-i2\pi ux} \, e^{-i 2\pi vy}$$

$$= (\cos 2\pi ux - i \sin 2\pi ux)(\cos 2\pi vy - i \sin 2\pi vy)$$

So once again we are viewing the Fourier transform as giving us the amplitude of the sine and cosine components of a signal at frequencies u and v.

Transform pairs

It is helpful to keep at hand some simple transform pairs, so that we don't always have to compute the Fourier integrals. First we will discuss 1-D pairs, then look at some 2-D relations which we will use for 2-D images.

Impulse. We have discussed briefly the impulse function $\delta(x)$, which has a non-zero value only at the origin. The δ-function has the sifting or sampling property defined by

$$\int_{-\infty}^{\infty} f(x)\, \delta(x - x')\, dx = f(x')$$

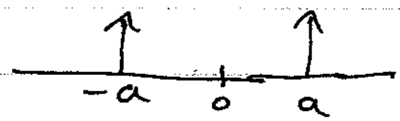
What is its Fourier transform, then?

$$F(s) = \int_{-\infty}^{\infty} \delta(x)\, e^{-i 2\pi s x}\, dx = e^{-i 2\pi s \cdot 0} = e^0 = \underline{\underline{1}}$$

In shorthand, if $f(x) \supset F(s)$ is read as $f(x)$ transforms into $F(s)$, then

$$\delta(x) \supset 1$$

Impulse pair. How about a pair of impulse at locations $+a$ and $-a$ on the axis, or



which we can represent as

$$f(x) = \delta(x+a) + \delta(x-a)$$

The transform is

$$F(s) = \int_{-\infty}^{\infty} [\delta(x+a) + \delta(x-a)]\, e^{-i2\pi sx}\, dx$$

$$= \int_{-\infty}^{\infty} \delta(x+a)\, e^{-i2\pi sx}\, dx + \int_{-\infty}^{\infty} \delta(x-a)\, e^{-i2\pi sx}\, dx$$

$$= e^{+i2\pi sa} + e^{-i2\pi sa}$$

$$= 2\cos 2\pi sa$$

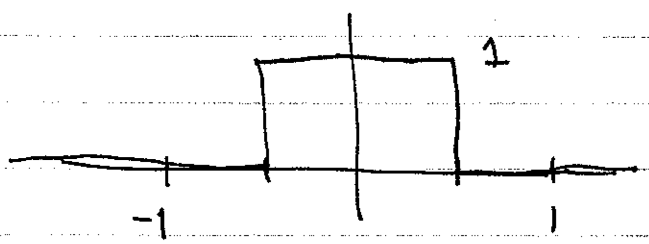
or

$$\delta(x+a) + \delta(x-a) \supset 2\cos 2\pi sa$$

Rect function. Another important function is the rect function, defined as

$$\mathrm{rect}(x) = 1 \quad \text{if } -\tfrac{1}{2} < x < \tfrac{1}{2}$$
$$= 0 \quad \text{otherwise}$$

A plot of the rect function looks like



Let's evaluate the Fourier transform.

$$F(s) = \int_{-\infty}^{\infty} \mathrm{rect}(x)\, e^{-i2\pi sx}\, dx$$

$$= \int_{-\frac{1}{2}}^{1/2} e^{-i2\pi sx}\, dx$$

$$= \left. \frac{e^{-i2\pi sx}}{-i2\pi s} \right|_{-\frac{1}{2}}^{1/2}$$

$$= \frac{e^{-i2\pi s\cdot\frac{1}{2}}}{-i2\pi s} - \frac{e^{+i2\pi s\cdot\frac{1}{2}}}{-i2\pi s}$$

$$= \frac{e^{-i\pi s}}{-i2\pi s} + \frac{e^{i\pi s}}{i2\pi s}$$

$$= \frac{e^{i\pi s} - e^{-i\pi s}}{i\,2\pi s}$$

Using again the Euler formula $e^{i\pi s} = \cos\pi s + i\sin\pi s$

$$= \frac{\cos\pi s + i\sin\pi s - \cos\pi s + i\sin\pi s}{i\,2\pi s}$$

$$= \frac{\sin\pi s}{\pi s}$$

This function shows up so often that we give it a special name, the sinc function:

$$\operatorname{sinc} s = \frac{\sin\pi\cdot s}{\pi\cdot s}$$

and $\qquad \operatorname{rect} x \supset \operatorname{sinc} s$

We could continue on but to make things short here is a list of important 1-D transforms. Don't memorize it but have it handy to refer to.

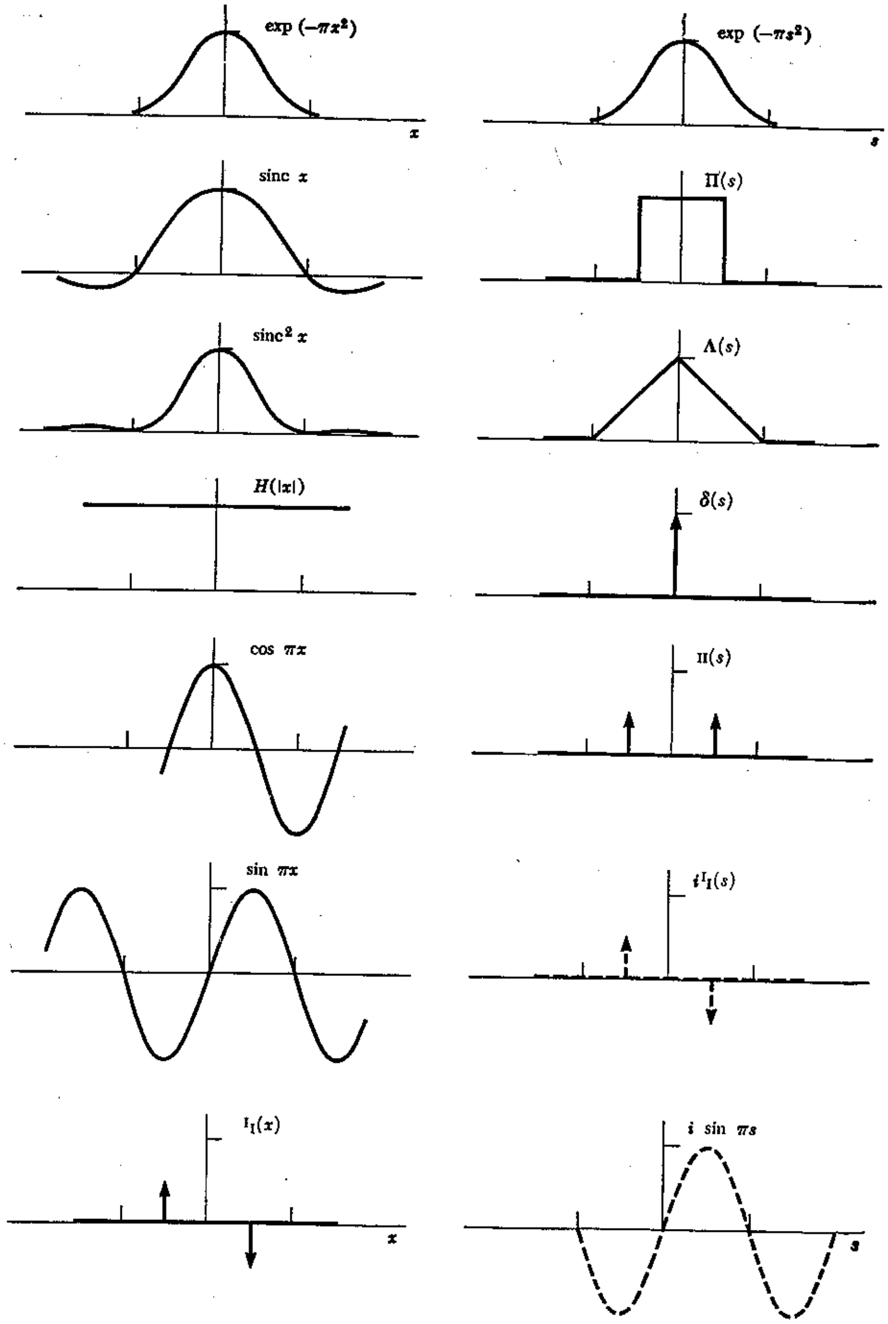


Fig. 6.1 Some Fourier transform pairs for reference.

All of these have inverse relations that differ from the forward relation at most by a sign change. For example,

$$e^{i\pi x} \supset \delta(s-\tfrac{1}{2})$$

while

$$\delta(x-\tfrac{1}{2}) \supset e^{-i\pi s}$$

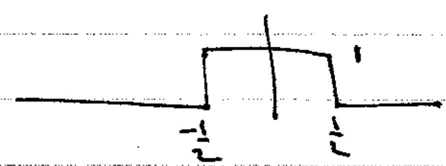
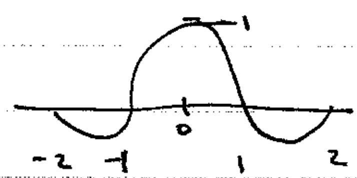
<u>Theorems</u>. There are many theorems that make it easy for us to manipulate the transforms of data. Some of the fundamental theorems are

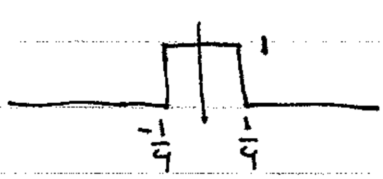
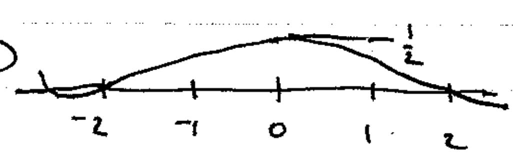
Addition: $f(x) + g(x) \supset F(s) + G(s)$

Similarity or scaling:

$$f(ax) \supset \frac{1}{|a|} F\left(\frac{s}{a}\right)$$

This theorem shows the ~~reciprocat~~ reciprocal relationship between "widths" of functions in the time and frequency domains. For example, consider the two functions rect(x) and rect(2x), and their transforms sinc(s) and $\frac{1}{2}\text{sinc}\left(\frac{s}{2}\right)$:

rect(x) =  $\supset$  = sinc x

rect(2x) =  $\supset$  $= \dfrac{\text{sinc}\,\frac{x}{2}}{2}$

The narrower function in the spatial domain transforms into the wider function in the frequency domain.

Shift theorem:

$$f(x-a) \supset e^{-i2\pi as} F(s)$$

Rayleigh's theorem:

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |F(s)|^2 \, ds$$

Notice that this theorem corresponds to conservation of energy.

Next: How about 2 dimensions?

In 2-D we can express the transform thusly, as we stated above:

$$F(u,v) = \iint_{-\infty}^{\infty} f(x,y) \, e^{-i2\pi(ux+vy)} \, dx \, dy$$

This definition implies a 2-D integration, which is much more computationally intensive than 1-D. If we had to integrate over 2-D surfaces every time we want to calculate a transform, we'd soon give up and go to other methods of analysis.

But we can simplify the above significantly if we realize that we can instead integrate one dimension at a time. To start with, break up the Fourier kernel into two parts:

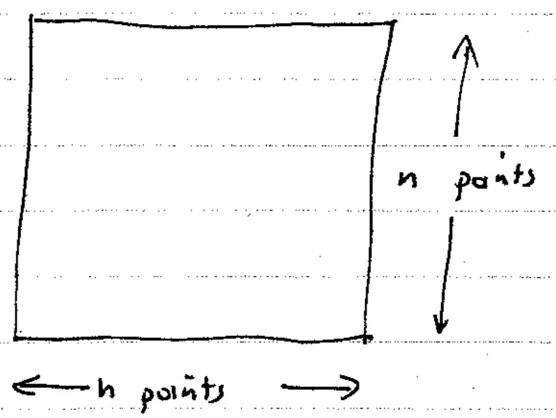
$$F(u,v) = \iint_{-\infty}^{\infty} f(x,y)\, e^{-i2\pi ux}\, e^{-i2\pi yv}\, dx\, dy$$

Rearranging terms,

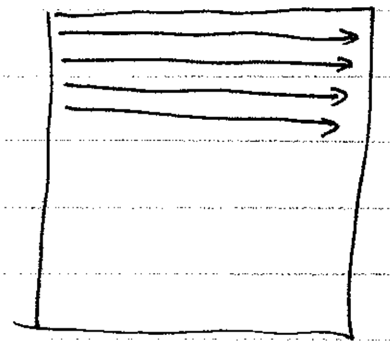
$$F(u,v) = \int_{-\infty}^{\infty} \underbrace{\int_{-\infty}^{\infty} f(x,y)\, e^{-i2\pi ux} dx}\; e^{-i2\pi vy}\, dy$$

This term is simply the 1-D transform of $f(x,y)$ in the x-direction. The 2-D transform is then the y-direction transform of the x-transform.

What does this mean computationally for us? Consider an image of size $n \times n$:

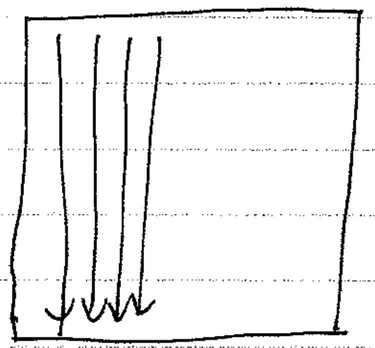


Straight numerical evaluation of the 2-D transform by integrating directly requires n^4 operations. Contrast this to successive 1-D transforms:



n 1-D transforms of n length across

followed by



n 1-D transforms of n length down

If a 1-D transform requires n^2 operations, the total needed is

$$n \cdot n^2 \cdot 2 = 2n^3$$

Thus the ratio of the 2-D approach to the successive 1-D approach is
$$\frac{n^4}{2n^3} = \frac{1}{2} n$$

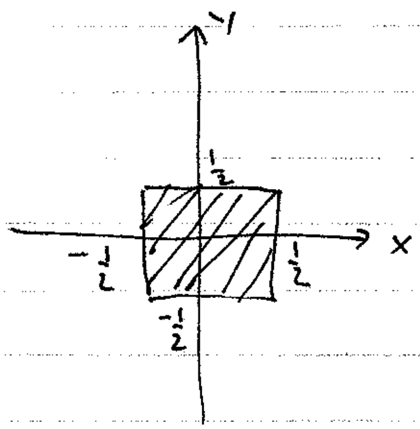
If your image is 1000 points on a side, it would take 500 times longer to do the 2-D integration!

Separability

A consequence of rewriting the 2-D transform as we did above is the separability theorem:

$$f(x)\, g(y) \supset F(u)\, G(v)$$

This theorem makes it easy for us to evaluate many 2-D transform pairs. Consider, for example, a square aperture



Note that this function can be represented as the product

$$\mathrm{rect}(x) \cdot \mathrm{rect}(y)$$

Thus its transform is directly

$$\mathrm{rect}(x)\, \mathrm{rect}(y) \supset \mathrm{sinc}\, u \; \mathrm{sinc}\, v$$

Other pairs follow by similar reasoning.