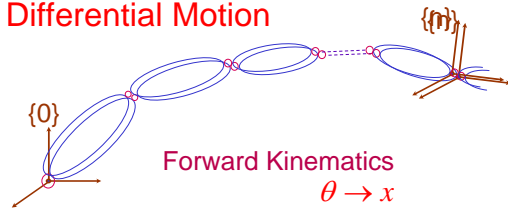


Instantaneous Kinematics

Differential Motion



Forward Kinematics
 $\theta \rightarrow x$

Instantaneous Kinematics
 $\theta + \delta\theta \rightarrow x + \delta x$

Relationship: $\delta\theta \leftrightarrow \delta x$

$\dot{\theta} \leftrightarrow \dot{x}$ ← Linear Velocity
 ← Angular Velocity

Joint Coordinates

coordinate $-i$: $\begin{cases} \theta_i & \text{revolute} \\ d_i & \text{prismatic} \end{cases}$

Joint coordinate $-i$: $q_i = \bar{\varepsilon}_i \theta_i + \varepsilon_i d_i$

with $\varepsilon_i = \begin{cases} 0 & \text{revolute} \\ 1 & \text{prismatic} \end{cases}$

and $\bar{\varepsilon}_i = 1 - \varepsilon_i$

Joint Coordinate Vector: $q = (q_1, q_2, \dots, q_n)^T$

Jacobians: Direct Differentiation

$$x = f(q); \quad \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} f_1(q) \\ f_2(q) \\ \vdots \\ f_m(q) \end{pmatrix}$$

$$\begin{aligned} \delta x_1 &= \frac{\partial f_1}{\partial q_1} \delta q_1 + \dots + \frac{\partial f_1}{\partial q_n} \delta q_n \\ &\vdots \\ \delta x_m &= \frac{\partial f_m}{\partial q_1} \delta q_1 + \dots + \frac{\partial f_m}{\partial q_n} \delta q_n \end{aligned} \quad \delta x = \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \dots & \frac{\partial f_1}{\partial q_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial q_1} & \dots & \frac{\partial f_m}{\partial q_n} \end{bmatrix} \delta q$$

$$\delta x_{(m \times 1)} = J_{(m \times n)}(q) \delta q_{(n \times 1)}$$

Jacobian

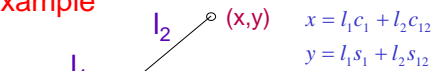
$$\delta x_{(m \times 1)} = J_{(m \times n)}(q) \delta q_{(n \times 1)}$$

$$\dot{x}_{(m \times 1)} = J_{(m \times n)}(q) \dot{q}_{(n \times 1)}$$

where

$$J_{ij}(q) = \frac{\partial}{\partial q_j} f_i(q)$$

Example



$$\begin{aligned} x &= l_1 c_1 + l_2 c_{12} \\ y &= l_1 s_1 + l_2 s_{12} \end{aligned}$$

$$\delta x = -(l_1 s_1 + l_2 s_{12}) \delta \theta_1 - l_2 s_{12} \delta \theta_2$$

$$\delta y = (l_1 c_1 + l_2 c_{12}) \delta \theta_1 + l_2 c_{12} \delta \theta_2$$

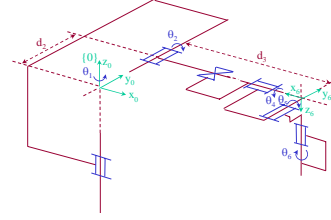
$$\delta X = \begin{bmatrix} \delta x \\ \delta y \end{bmatrix} = \begin{bmatrix} -y & -l_2 s_{12} \\ x & l_2 c_{12} \end{bmatrix} \begin{bmatrix} \delta \theta_1 \\ \delta \theta_2 \end{bmatrix}$$

$$\delta x = J(\theta) \delta \theta$$

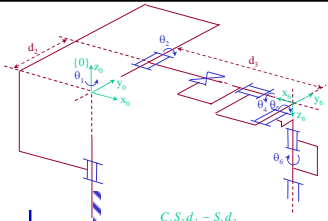
$$\dot{x} = J(\theta) \dot{\theta}$$

$$J \equiv \begin{bmatrix} \frac{\partial x}{\partial \theta_1} & \frac{\partial x}{\partial \theta_2} \\ \frac{\partial y}{\partial \theta_1} & \frac{\partial y}{\partial \theta_2} \end{bmatrix} = \begin{bmatrix} -y & -l_2 s_{12} \\ x & l_2 c_{12} \end{bmatrix}$$

Stanford Scheinman Arm



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	-90	0	d_1	θ_2
3	90	0	d_2	θ_3
4	0	0	d_3	θ_4
5	-90	0	0	θ_5
6	90	0	0	θ_6



$$x = \begin{pmatrix} x_p \\ r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} C_1 S_2 d_3 - S_1 d_2 \\ S_1 S_2 d_3 + C_1 d_2 \\ C_2 d_3 \\ C_1 [C_2 (C_1 C_3 C_6 - S_1 S_6) - S_2 S_3 C_6] - S_1 (S_2 C_3 C_6 + C_2 S_3) \\ S_1 [C_2 (C_1 C_3 C_6 - S_1 S_6) - S_2 S_3 C_6] + C_1 (S_2 C_3 C_6 + C_2 S_3) \\ -S_1 (C_1 C_3 C_6 - S_1 S_6) - C_2 S_3 C_6 \\ C_1 [-C_2 (C_1 C_3 S_6 + S_1 C_6) + S_2 S_3 S_6] - S_1 (-S_2 C_3 S_6 + C_2 C_6) \\ S_1 [-C_2 (C_1 C_3 S_6 + S_1 C_6) + S_2 S_3 S_6] + C_1 (-S_2 C_3 S_6 + C_2 C_6) \\ S_2 (C_2 C_3 S_6 + S_1 C_6) + C_2 S_3 S_6 \\ C_1 (C_2 C_3 S_6 + S_1 C_6) - S_1 S_3 S_6 \\ S_1 (C_2 C_3 S_6 + S_1 C_6) + C_1 S_3 S_6 \\ -S_1 C_3 S_6 + C_2 C_3 \end{pmatrix}$$

Stanford Scheinman Arm

Position

$$x_p = \begin{bmatrix} c_1 s_2 d_3 - s_1 d_2 \\ s_1 s_2 d_3 + c_1 d_2 \\ c_2 d_3 \end{bmatrix}$$

$$\dot{x}_p = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{bmatrix} \dots \\ \dots \\ \dots \\ \dots \\ \dots \\ \dots \end{bmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{pmatrix}$$

$$\dot{x}_p(3 \times 1) = J_{x_p(3 \times 6)}(q) \dot{q}(6 \times 1)$$

Linear Velocity \mathbf{V}

Orientation: Direction Cosines

$$\dot{x}_R = J_{x_R}(q) \dot{q} \quad x_R = \begin{bmatrix} r_1(q) \\ r_2(q) \\ r_3(q) \end{bmatrix}$$

$$\dot{x}_R = \begin{pmatrix} \dot{r}_1 \\ \dot{r}_2 \\ \dot{r}_3 \end{pmatrix}_{(9 \times 1)} = \begin{pmatrix} \frac{\partial r_1}{\partial q_1} & \dots & \frac{\partial r_1}{\partial q_6} \\ \frac{\partial r_2}{\partial q_1} & \dots & \frac{\partial r_2}{\partial q_6} \\ \frac{\partial r_3}{\partial q_1} & \dots & \frac{\partial r_3}{\partial q_6} \end{pmatrix}_{(9 \times 6)} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_6 \end{pmatrix}_{(6 \times 1)}$$

$$x_R = \begin{bmatrix} C_1 [C_2 (C_1 C_3 C_6 - S_1 S_6) - S_2 S_3 C_6] - S_1 (S_2 C_3 C_6 + C_2 S_3) \\ S_1 [C_2 (C_1 C_3 C_6 - S_1 S_6) - S_2 S_3 C_6] + C_1 (S_2 C_3 C_6 + C_2 S_3) \\ -S_1 (C_1 C_3 C_6 - S_1 S_6) - C_2 S_3 C_6 \\ C_1 [-C_2 (C_1 C_3 S_6 + S_1 C_6) + S_2 S_3 S_6] - S_1 (-S_2 C_3 S_6 + C_2 C_6) \\ S_1 [-C_2 (C_1 C_3 S_6 + S_1 C_6) + S_2 S_3 S_6] + C_1 (-S_2 C_3 S_6 + C_2 C_6) \\ S_2 (C_2 C_3 S_6 + S_1 C_6) + C_2 S_3 S_6 \\ C_1 (C_2 C_3 S_6 + S_1 C_6) - S_1 S_3 S_6 \\ S_1 (C_2 C_3 S_6 + S_1 C_6) + C_1 S_3 S_6 \\ -S_1 C_3 S_6 + C_2 C_3 \end{bmatrix}$$

$$\dot{x}_R = \begin{pmatrix} \dot{r}_1 \\ \dot{r}_2 \\ \dot{r}_3 \end{pmatrix}_{(9 \times 1)} = \begin{pmatrix} \frac{\partial r_1}{\partial q_1} & \dots & \frac{\partial r_1}{\partial q_6} \\ \frac{\partial r_2}{\partial q_1} & \dots & \frac{\partial r_2}{\partial q_6} \\ \frac{\partial r_3}{\partial q_1} & \dots & \frac{\partial r_3}{\partial q_6} \end{pmatrix}_{(9 \times 6)} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_6 \end{pmatrix}_{(6 \times 1)}$$

Representations

$$X = \begin{bmatrix} x_P \\ x_R \end{bmatrix}$$

- Cartesian
- Spherical
- Cylindrical
- ...
- Euler Angles
- Direction Cosines
- Euler Parameters

Jacobian for X

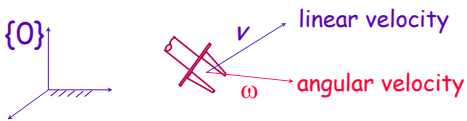
$$\begin{aligned} \dot{x}_P &= J_{X_P}(q) \dot{q} \\ \dot{x}_R &= J_{X_R}(q) \dot{q} \end{aligned} \quad \begin{pmatrix} \dot{x}_P \\ \dot{x}_R \end{pmatrix} = \begin{pmatrix} J_{X_P}(q) \\ J_{X_R}(q) \end{pmatrix} \dot{q}$$

Cartesian & Direction Cosines

$$\dot{X}_{(12 \times 1)} = J_X(q)_{(12 \times 6)} \dot{q}_{(6 \times 1)}$$

The Jacobian is dependent on the representation

Basic Jacobian



$$\begin{pmatrix} v \\ \omega \end{pmatrix}_{(6 \times 1)} = J_0(q)_{(6 \times n)} \dot{q}_{(n \times 1)}$$

$$\dot{x}_P = E_P(x_P) v$$

$$\dot{x}_R = E_R(x_R) \omega$$

Examples

$$\star x_R = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}; E_R(x_R) = \begin{pmatrix} -\frac{s\alpha \cdot c\beta}{s\beta} & \frac{c\alpha \cdot c\beta}{s\beta} & 1 \\ c\alpha & s\alpha & 0 \\ \frac{s\alpha}{s\beta} & -\frac{c\alpha}{s\beta} & 0 \end{pmatrix}$$

$$\star x_P = \begin{pmatrix} x \\ y \\ z \end{pmatrix}; E_P(x_P) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Jacobian for X

Given a representation $x = \begin{bmatrix} x_P \\ x_R \end{bmatrix}$

$$\dot{x} = J_x(q) \dot{q}$$

$$J_x(q) = E(x) J_0(q)$$

Basic Jacobian $\begin{pmatrix} v \\ \omega \end{pmatrix} = J_0(q) \dot{q}$

Jacobian and Basic Jacobian

$$\begin{pmatrix} v \\ \omega \end{pmatrix} = J_0(q) \dot{q}$$

$$\begin{pmatrix} v \\ \omega \end{pmatrix} = \begin{pmatrix} J_v \\ J_\omega \end{pmatrix} \dot{q}$$

$$\begin{cases} v = J_v \cdot \dot{q} \\ \omega = J_\omega \cdot \dot{q} \end{cases}$$

$$\dot{x}_P = E_P \cdot v \Rightarrow \dot{x}_P = (E_P \cdot J_v) \dot{q}$$

$$\dot{x}_R = E_R \cdot \omega \Rightarrow \dot{x}_R = (E_R \cdot J_\omega) \dot{q}$$

$$\begin{cases} J_{X_P} = E_P \cdot J_v \\ J_{X_R} = E_R \cdot J_\omega \end{cases}$$

$$J = \begin{pmatrix} J_{XP} \\ J_{XR} \end{pmatrix} = \begin{pmatrix} E_P & 0 \\ 0 & E_R \end{pmatrix} \begin{pmatrix} J_v \\ J_w \end{pmatrix}$$

$$\underline{J(q)} = \underline{E(X)} \quad \underline{J_0(q)}$$

$$\underline{\begin{pmatrix} v \\ w \end{pmatrix}} = \underline{J_0(q)} \dot{q}$$

With Cartesian Coordinates

$$E_P = I_3; J_{XP} = J_v; \text{ and } E = \begin{pmatrix} I & 0 \\ 0 & E_R \end{pmatrix}$$

Position Representations

Cartesian Coordinates (x, y, z)

$$E_p(X) = I_3$$

Cylindrical Coordinates (ρ, θ, z)

Using $(x \ y \ z)^T = (\rho \cos \theta \ \rho \sin \theta \ z)^T$

$$E_p(X) = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta / \rho & \cos \theta / \rho & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Spherical Coordinates (ρ, θ, ϕ)

Using

$$(x \ y \ z)^T = (\rho \cos \theta \sin \phi \ \rho \sin \theta \sin \phi \ \rho \cos \theta)^T$$

$$E_p(X) = \begin{pmatrix} \cos \theta \sin \phi & \sin \theta \sin \phi & \cos \phi \\ -\sin \theta / (\rho \sin \phi) & \cos \theta / (\rho \sin \phi) & 0 \\ \cos \theta \cos \phi / \rho & \sin \theta \cos \phi / \rho & -\sin \phi / \rho \end{pmatrix}$$

Euler Angles

$$x_R = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}; E_R(x_R) = \begin{pmatrix} -\frac{s\alpha.c\beta}{s\beta} & \frac{c\alpha.c\beta}{s\beta} & 1 \\ c\alpha & s\alpha & 0 \\ \frac{s\alpha}{s\beta} & -\frac{c\alpha}{s\beta} & 0 \end{pmatrix}$$

Singularity of the representation
for $\beta = k\pi$

Jacobian for X

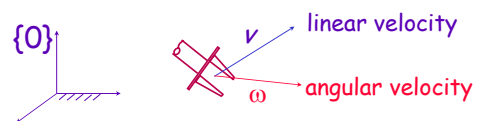
Given a representation $x = \begin{bmatrix} x_P \\ x_R \end{bmatrix}$

$$\dot{x} = J_x(q) \dot{q}$$

$$J_x(q) = E(x) J_0(q)$$

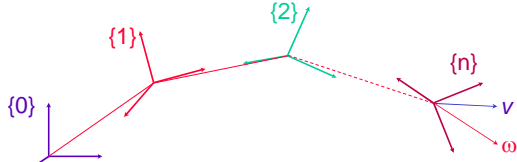
Basic Jacobian $\begin{pmatrix} v \\ w \end{pmatrix} = J_0(q) \dot{q}$

Basic Jacobian



$$\begin{pmatrix} v \\ \omega \end{pmatrix}_{(6 \times 1)} = J(q)_{(6 \times n)} \dot{q}_{(n \times 1)}$$

Spatial Mechanisms

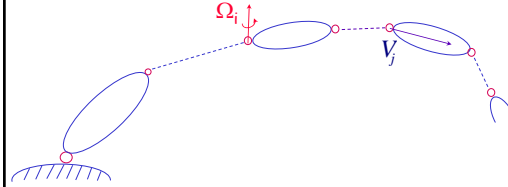


Propagation of velocities

$$\dot{x} \begin{cases} v : \text{linear velocity} \\ \omega : \text{angular velocity} \end{cases}$$

$$\dot{x} = J(\theta) \cdot \dot{\theta}$$

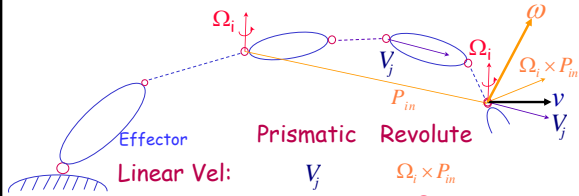
The Jacobian (EXPLICIT FORM)



$$\text{Revolute Joint } \Omega_i = Z_i \dot{q}_i$$

$$\text{Prismatic Joint } V_i = Z_i \dot{q}_i$$

The Jacobian (EXPLICIT FORM)



Linear Vel: V_j

Angular Vel: none

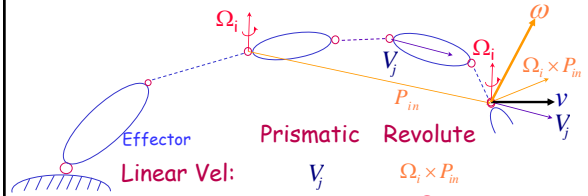
Effector Linear Velocity

$$v = \sum_{i=1}^n [\epsilon_i V_i + \bar{\epsilon}_i (\Omega_i \times P_{in})] \iff V_i = Z_i \dot{q}_i$$

Effector Angular Velocity

$$\omega = \sum_{i=1}^n \bar{\epsilon}_i \Omega_i \iff \Omega_i = Z_i \dot{q}_i$$

The Jacobian (EXPLICIT FORM)



Linear Vel: V_j

Angular Vel: none

Effector Linear Velocity

$$v = \sum_{i=1}^n [\epsilon_i Z_i + \bar{\epsilon}_i (Z_i \times P_{in})] \dot{q}_i \iff V_i = Z_i \dot{q}_i$$

Effector Angular Velocity

$$\omega = \sum_{i=1}^n (\bar{\epsilon}_i Z_i) \dot{q}_i \iff \Omega_i = Z_i \dot{q}_i$$

$$v = [\epsilon_1 Z_1 + \bar{\epsilon}_1 (Z_1 \times P_{1n})] \dot{q}_1 + \dots + [\epsilon_{n-1} Z_{n-1} + \bar{\epsilon}_{n-1} (Z_{n-1} \times P_{(n-1)n})] \dot{q}_{n-1} + \epsilon_n Z_n \dot{q}_n$$

$$v = [\epsilon_1 Z_1 + \bar{\epsilon}_1 (Z_1 \times P_{1n}) \quad \epsilon_2 Z_2 + \bar{\epsilon}_2 (Z_2 \times P_{2n}) \quad \dots] \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

$$v = J_v \dot{q}$$

$$\omega = \bar{\epsilon}_1 Z_1 \dot{q}_1 + \bar{\epsilon}_2 Z_2 \dot{q}_2 + \dots + \bar{\epsilon}_n Z_n \dot{q}_n$$

$$\omega = [\bar{\epsilon}_1 Z_1 \quad \bar{\epsilon}_2 Z_2 \quad \dots \quad \bar{\epsilon}_n Z_n] \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

$$\omega = J_\omega \dot{q}$$

The Jacobian

$$J = \begin{pmatrix} J_v \\ J_\omega \end{pmatrix}$$

Matrix J_v (direct differentiation)

$$v = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \dot{x}_p = \frac{\partial x_p}{\partial q_1} \dot{q}_1 + \frac{\partial x_p}{\partial q_2} \dot{q}_2 + \dots + \frac{\partial x_p}{\partial q_n} \dot{q}_n$$

$$J_v = \begin{pmatrix} \frac{\partial x_p}{\partial q_1} & \frac{\partial x_p}{\partial q_2} & \dots & \frac{\partial x_p}{\partial q_n} \end{pmatrix}$$

Jacobian in a Frame

Vector Representation

$$J = \begin{pmatrix} \frac{\partial x_P}{\partial q_1} & \frac{\partial x_P}{\partial q_2} & \dots & \frac{\partial x_P}{\partial q_n} \\ \bar{\epsilon}_1 \cdot Z_1 & \bar{\epsilon}_2 \cdot Z_2 & \dots & \bar{\epsilon}_n \cdot Z_n \end{pmatrix}$$

In {0}

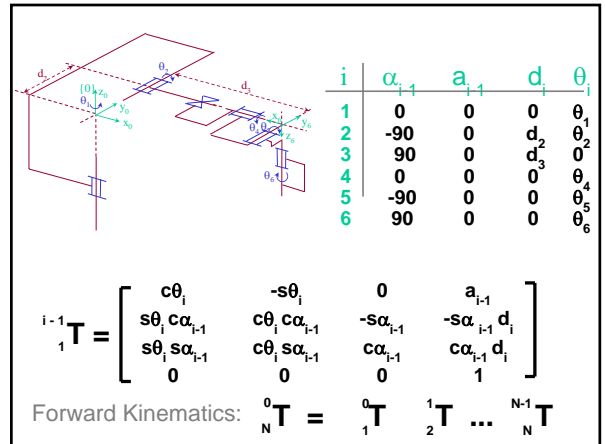
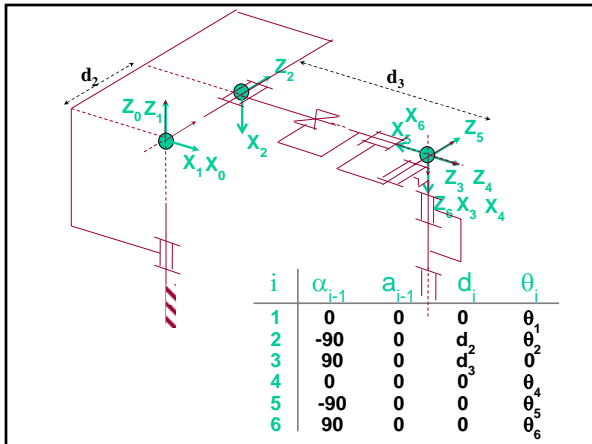
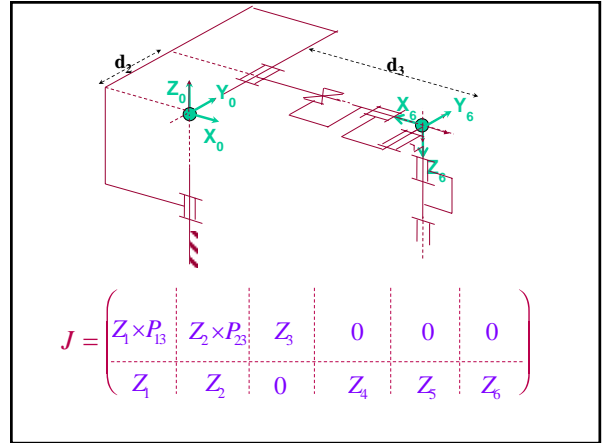
$${}^0J = \begin{pmatrix} \frac{\partial {}^0x_P}{\partial q_1} & \frac{\partial {}^0x_P}{\partial q_2} & \dots & \frac{\partial {}^0x_P}{\partial q_n} \\ \bar{\epsilon}_1 \cdot {}^0Z_1 & \bar{\epsilon}_2 \cdot {}^0Z_2 & \dots & \bar{\epsilon}_n \cdot {}^0Z_n \end{pmatrix}$$

J in Frame {0}

$${}^0Z_i = {}^0R \cdot {}^iZ_i; \quad {}^iZ_i = Z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$${}^0J = \begin{pmatrix} \frac{\partial ({}^0x_P)}{\partial q_1} & \frac{\partial ({}^0x_P)}{\partial q_2} & \dots & \frac{\partial ({}^0x_P)}{\partial q_n} \\ \bar{\epsilon}_1 \cdot ({}^0R \cdot Z) & \bar{\epsilon}_2 \cdot ({}^0R \cdot Z) & \dots & \bar{\epsilon}_n \cdot ({}^0R \cdot Z) \end{pmatrix}$$

Stanford Scheinman Arm



Stanford Scheinman Arm

$${}^0_1T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^1_2T = \begin{bmatrix} c_2 & -s_2 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ -s_2 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^2_3T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -d_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^3_4T = \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^4_5T = \begin{bmatrix} c_5 & -s_5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s_5 & -c_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^5_6T = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0_1T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0_2T = \begin{bmatrix} c_1c_2 & -c_1s_2 & -s_1 & -s_1d_2 \\ s_1c_2 & -s_1s_2 & c_1 & c_1d_2 \\ -s_2 & -c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0_3T = \begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 & c_1d_3s_2 - s_1d_2 \\ s_1c_2 & c_1 & s_1s_2 & s_1d_3s_2 + c_1d_2 \\ -s_2 & 0 & c_2 & d_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0_4T = \begin{bmatrix} c_1c_2c_4 - s_1s_4 & -c_1c_2s_4 - s_1c_4 & c_1s_2 & c_1d_3s_2 - s_1d_2 \\ s_1c_2c_4 + c_1s_4 & -s_1c_2s_4 + c_1c_4 & s_1s_2 & s_1d_3s_2 + c_1d_2 \\ -s_2c_4 & s_2s_4 & c_2 & d_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0_5T = \begin{bmatrix} X & X & -c_1c_2s_4 - s_1c_4 & c_1d_3s_2 - s_1d_2 \\ X & X & -s_1c_2s_4 + c_1c_4 & s_1d_3s_2 + c_1d_2 \\ X & X & s_2s_4 & d_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0_6T = \begin{bmatrix} X & X & c_1c_2c_4s_5 - s_1s_4s_5 + c_1s_2s_5 & c_1d_3s_2 - s_1d_2 \\ X & X & s_1c_2c_4s_5 + c_1s_4s_5 + s_1s_2c_5 & s_1d_3s_2 + c_1d_2 \\ X & X & -s_2c_4s_5 + c_2c_5 & d_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0_6T = \begin{bmatrix} X & X & c_1c_2c_4s_5 - s_1s_4s_5 + c_1s_2s_5 & c_1d_3s_2 - s_1d_2 \\ X & X & s_1c_2c_4s_5 + c_1s_4s_5 + s_1s_2c_5 & s_1d_3s_2 + c_1d_2 \\ X & X & -s_2c_4s_5 + c_2c_5 & d_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$x = \begin{pmatrix} x_p \\ r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{bmatrix} c_1s_2d_3 - s_1d_2 \\ s_1s_2d_3 + c_1d_2 \\ c_2d_3 \\ c_1[C_2(C_1C_3C_4 - S_2S_4) - S_1S_2C_4] - S_1(S_1C_2C_4 + C_2S_4) \\ S_1[C_2(C_1C_3C_4 - S_2S_4) - S_2S_2C_4] + C_1(S_1C_2C_4 + C_2S_4) \\ -S_2(C_1C_3C_4 - S_2S_4) - C_2S_2C_4 \\ c_1[-C_2(C_1C_3C_4 + S_2S_4) + S_2S_2S_4] - S_1(-S_1C_2C_4 + C_2C_4) \\ S_1[-C_2(C_1C_3C_4 + S_2S_4) + S_2S_2S_4] + C_1(-S_1C_2C_4 + C_2C_4) \\ S_2(C_1C_3C_4 + S_2C_4) + C_2S_2S_4 \\ C_1(C_2C_3S_4 + S_2C_2) - S_1S_1S_4 \\ S_1(C_2C_3S_4 + S_2C_2) + C_1S_1S_4 \\ -S_2C_1C_3 + C_2C_1 \end{bmatrix}$$

Stanford Scheinman Arm Jacobian

$${}^0J = \begin{pmatrix} \frac{\partial^0 x_p}{\partial q_1} & \frac{\partial^0 x_p}{\partial q_2} & \frac{\partial^0 x_p}{\partial q_3} & 0 & 0 & 0 \\ {}^0Z_1 & {}^0Z_2 & 0 & {}^0Z_4 & {}^0Z_5 & {}^0Z_6 \end{pmatrix}$$

$$\begin{bmatrix} -c_1d_2 - s_1s_2d_3 & c_1c_2d_3 & c_1s_2 & 0 & 0 & 0 \\ -s_1d_2 + c_1s_2d_3 & s_1c_2d_3 & s_1s_2 & 0 & 0 & 0 \\ 0 & -s_2d_3 & c_2 & 0 & 0 & 0 \\ 0 & -s_1 & 0 & c_1s_2 & -c_1c_2s_4 - s_1c_4 & c_1c_2c_4s_5 - s_1s_4s_5 + c_1s_2c_5 \\ 0 & c_1 & 0 & s_1s_2 & -s_1c_2s_4 + c_1c_4 & s_1c_2c_4s_5 + c_1s_4s_5 + s_1s_2c_5 \\ 1 & 0 & 0 & c_2 & s_2s_4 & -s_2c_4s_5 + c_2c_5 \end{bmatrix}$$

Kinematic Singularity

The Effector Locality loses the ability to move in a direction or to rotate about a direction - singular direction

$$J = (J_1 \ J_2 \ \dots \ J_n)$$

$$\det(J) = 0$$

$$\det({}^i J) = \det({}^j J)$$

Kinematic Singularity

$${}^B J = \begin{pmatrix} {}^B R & 0 \\ 0 & {}^B R \end{pmatrix} {}^A J$$

$$\det[{}^B J] \equiv \det[{}^A J]$$

$$\det({}^i J) = \det({}^j J)$$

Singular Configurations

$$\det[J(q)] = 0$$

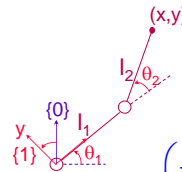
⇒ Singular Configurations

$$\det[J(q)] = S_1(q)S_2(q)\dots S_s(q) = 0$$



$$\begin{matrix} S_1(q) = 0 \\ S_2(q) = 0 \\ \vdots \\ S_s(q) = 0 \end{matrix}$$

Example (Kinematic Singularities)



$$x = l_1 C1 + l_2 C12$$

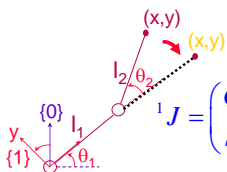
$$y = l_1 S1 + l_2 S12$$

$$J = \begin{pmatrix} -(l_1 S1 + l_2 S12) & -l_2 S12 \\ l_1 C1 + l_2 C12 & l_2 C12 \end{pmatrix}$$

$$\det(J) = l_1 l_2 S2$$

Singularity at $q_2 = k\pi$

Example (Kinematic Singularities)



$${}^1 J = {}^1 R \ {}^0 J$$

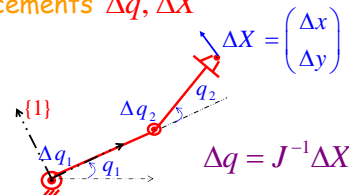
$${}^1 J = \begin{pmatrix} C1 & -S1 \\ S1 & C1 \end{pmatrix} \begin{pmatrix} -l_2 S2 & -l_2 S2 \\ l_1 + l_2 C2 & l_2 C2 \end{pmatrix}$$

At Singularity

$${}^1 J = \begin{pmatrix} 0 & 0 \\ l_1 + l_2 & l_2 \end{pmatrix}$$

$$\begin{matrix} {}^1 \delta x = 0 \\ {}^1 \delta y = (l_1 + l_2)\delta\theta_1 + l_2\delta\theta_2 \end{matrix}$$

Small Displacements $\Delta q, \Delta X$

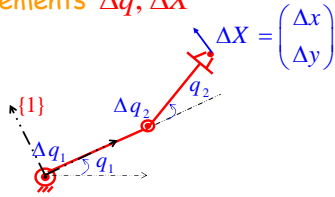


$$\Delta q = J^{-1} \Delta X$$

small θ_2

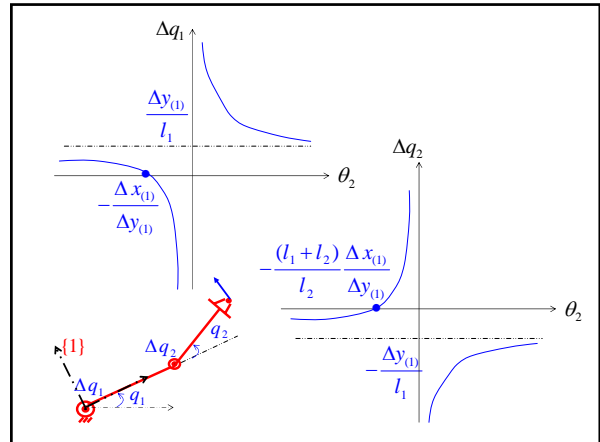
$$J_{(0)}^{-1} \cong \begin{pmatrix} \frac{1}{l_1 \theta_2} & \frac{1}{l_1} \\ -\frac{1}{l_1 + l_2} & -\frac{1}{l_1} \end{pmatrix}$$

Small Displacements $\Delta q, \Delta X$

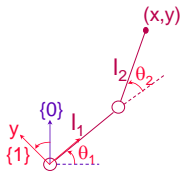


$$\Delta q_1 = \frac{\Delta x_{(1)}}{l_1} \cdot \frac{1}{\theta_2} + \frac{\Delta y_{(1)}}{l_1}$$

$$\Delta q_2 = \frac{(l_1 + l_2) \Delta x_{(1)}}{l_1 l_2} \cdot \frac{1}{\theta_2} + \frac{\Delta y_{(1)}}{l_1}$$



Kinematic Singularities (reduced matrix)

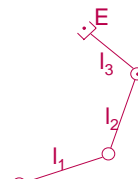


$$J = \begin{bmatrix} -(l_1 s_1 + l_2 s_12) & -l_2 s_12 \\ l_1 c_1 + l_2 c_12 & l_2 c_12 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\det(J) = l_1 l_2 s_2$$

$$J = \begin{bmatrix} -(l_1 s_1 + l_2 s_12) & -l_2 s_12 \\ l_1 c_1 + l_2 c_12 & l_2 c_12 \end{bmatrix}$$

Singularity at $q_2 = k\pi$

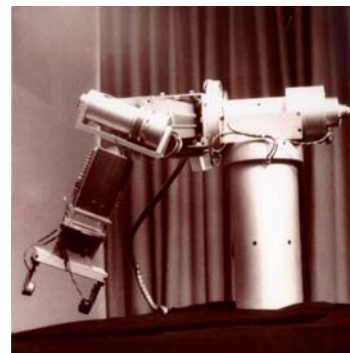


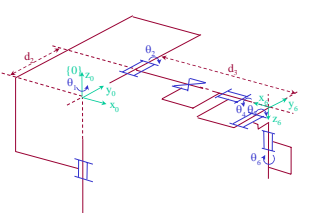
$${}^0 J_E = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} - l_3 s_{123} & -l_2 s_{12} - l_3 s_{123} & -l_3 s_{123} \\ l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{123} & l_3 c_{123} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$${}^0 J_E = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} - l_3 s_{123} & -l_2 s_{12} - l_3 s_{123} & -l_3 s_{123} \\ l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{123} & l_3 c_{123} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$${}^0 J_E = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} - l_3 s_{123} & -l_2 s_{12} - l_3 s_{123} & -l_3 s_{123} \\ l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{123} & l_3 c_{123} \\ 1 & 1 & 1 \end{bmatrix}$$

Stanford Scheinman Arm





i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	-90	0	0	θ_2
3	90	0	d_3	θ_3
4	0	0	0	θ_4
5	-90	0	0	θ_5
6	90	0	0	θ_6

$${}^{i-1}\mathbf{T} = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Forward Kinematics: ${}^0\mathbf{T} = {}^0\mathbf{T}_1 {}^1\mathbf{T}_2 \dots {}^{N-1}\mathbf{T}_N$

Stanford Scheinman Arm Jacobian

$${}^0\mathbf{J} = \begin{pmatrix} \frac{\partial^0 x_P}{\partial q_1} & \frac{\partial^0 x_P}{\partial q_2} & \frac{\partial^0 x_P}{\partial q_3} & 0 & 0 & 0 \\ \frac{\partial^0 y_P}{\partial q_1} & \frac{\partial^0 y_P}{\partial q_2} & \frac{\partial^0 y_P}{\partial q_3} & 0 & 0 & 0 \\ \frac{\partial^0 z_P}{\partial q_1} & \frac{\partial^0 z_P}{\partial q_2} & \frac{\partial^0 z_P}{\partial q_3} & 0 & 0 & 0 \end{pmatrix}$$

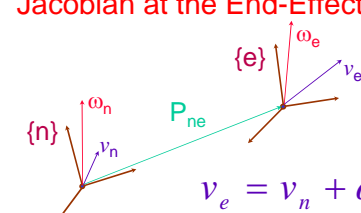
$$\begin{bmatrix} -c_1 d_2 - s_1 s_2 d_3 & c_1 c_2 d_3 & c_1 s_2 & 0 & 0 & 0 \\ -s_1 d_2 + c_1 s_2 d_3 & s_1 c_2 d_3 & s_1 s_2 & 0 & 0 & 0 \\ 0 & -s_2 d_3 & c_2 & 0 & 0 & 0 \\ 0 & -s_1 & 0 & c_1 s_2 & -c_1 c_2 s_4 - s_1 c_4 & c_1 c_2 c_4 s_5 - s_1 s_4 s_5 + c_1 s_2 c_5 \\ 0 & c_1 & 0 & s_1 s_2 & -s_1 c_2 s_4 + c_1 c_4 & s_1 c_2 c_4 s_5 + c_1 s_4 s_5 + s_1 s_2 c_5 \\ 1 & 0 & 0 & c_2 & s_2 s_4 & -s_2 c_4 s_5 + c_5 c_2 \end{bmatrix}$$

Stanford Scheinman Arm Jacobian

$\theta_5 = k\pi$

$$\mathbf{J} = \begin{bmatrix} -c_1 d_2 - s_1 s_2 d_3 & c_1 c_2 d_3 & c_1 s_2 & 0 & 0 & 0 \\ -s_1 d_2 + c_1 s_2 d_3 & s_1 c_2 d_3 & s_1 s_2 & 0 & 0 & 0 \\ 0 & -s_2 d_3 & c_2 & 0 & 0 & 0 \\ 0 & -s_1 & 0 & c_1 s_2 & -c_1 c_2 s_4 - s_1 c_4 & c_1 s_2 \\ 0 & c_1 & 0 & s_1 s_2 & -s_1 c_2 s_4 + c_1 c_4 & s_1 s_2 \\ 1 & 0 & 0 & c_2 & s_2 s_4 & c_2 \end{bmatrix}$$

Jacobian at the End-Effector



$$\mathbf{v}_e = \mathbf{v}_n + \boldsymbol{\omega}_n \times \mathbf{P}_{ne}$$

$$\begin{cases} \mathbf{v}_e = \mathbf{v}_n - \mathbf{P}_{ne} \times \boldsymbol{\omega}_n \\ \boldsymbol{\omega}_e = \boldsymbol{\omega}_n \end{cases}$$

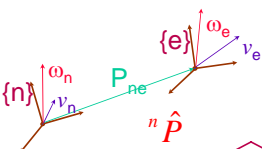
$$\begin{cases} \mathbf{v}_e = \mathbf{v}_n - \mathbf{P}_{ne} \times \boldsymbol{\omega}_n \\ \boldsymbol{\omega}_e = \boldsymbol{\omega}_n \end{cases}$$

$$\begin{pmatrix} \mathbf{v}_e \\ \boldsymbol{\omega}_e \end{pmatrix} = \begin{pmatrix} \mathbf{I} & -\hat{\mathbf{P}}_{ne} \\ \mathbf{O} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{v}_n \\ \boldsymbol{\omega}_n \end{pmatrix}$$

$$\mathbf{J}_e \dot{\mathbf{q}} = \begin{pmatrix} \mathbf{I} & -\hat{\mathbf{P}}_{ne} \\ \mathbf{O} & \mathbf{I} \end{pmatrix} \mathbf{J}_n \dot{\mathbf{q}}$$

$$\mathbf{J}_e = \begin{pmatrix} \mathbf{I} & -\hat{\mathbf{P}}_{ne} \\ \mathbf{O} & \mathbf{I} \end{pmatrix} \mathbf{J}_n$$

Cross Product Operator (in diff. frames)



$${}^0\mathbf{J}_e = \begin{pmatrix} \mathbf{I} & -{}^0\hat{\mathbf{P}}_{ne} \\ \mathbf{O} & \mathbf{I} \end{pmatrix} {}^0\mathbf{J}_n$$

$${}^0\hat{\mathbf{P}} \neq {}^0\mathbf{R} {}^n\hat{\mathbf{P}}; \quad \widehat{{}^0\mathbf{P}} = (\widehat{{}^0\mathbf{R} {}^n\mathbf{P}}) \neq {}^0\mathbf{R} \cdot \widehat{{}^n\mathbf{P}}$$

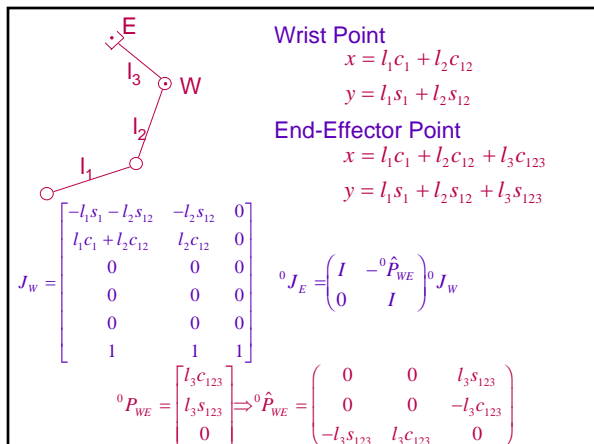
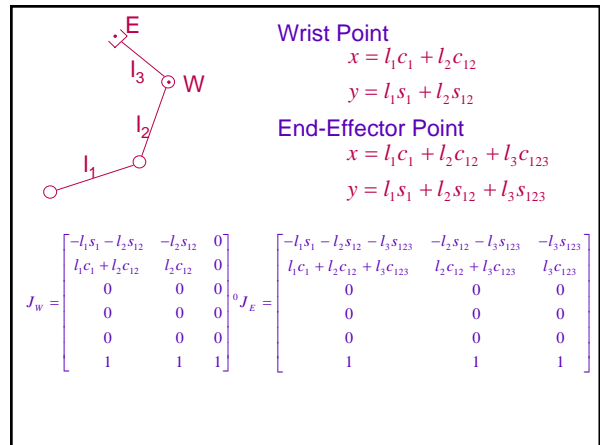
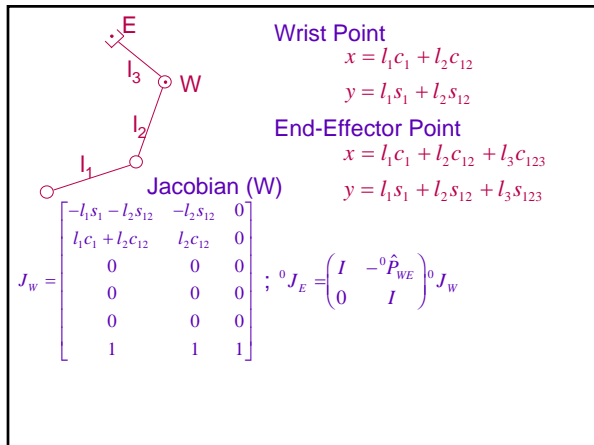
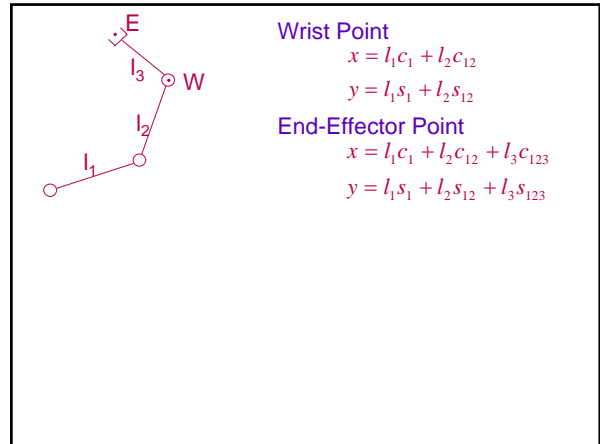
$${}^0\mathbf{P} \times {}^0\boldsymbol{\omega} = {}^0\mathbf{R} \cdot ({}^n\mathbf{P} \times {}^n\boldsymbol{\omega})$$

$${}^0\hat{\mathbf{P}} \cdot {}^0\boldsymbol{\omega} = {}^0\mathbf{R} \cdot ({}^n\hat{\mathbf{P}} \cdot {}^n\boldsymbol{\omega}) = {}^0\mathbf{R} \cdot ({}^n\hat{\mathbf{P}} \cdot {}^0\mathbf{R}^T \cdot {}^0\boldsymbol{\omega})$$

$$\boxed{{}^0\hat{\mathbf{P}} = {}^0\mathbf{R} {}^n\hat{\mathbf{P}} {}^0\mathbf{R}^T}$$

$${}^i J = \begin{pmatrix} {}^i R & 0 \\ 0 & {}^j R \end{pmatrix} {}^j J$$

$${}^0 J_e = \begin{pmatrix} {}^0 R & -{}^0 R {}^n \hat{P}_{ne} {}^0 R^T \\ 0 & {}^n R \end{pmatrix} {}^n J_n$$



Resolved Motion Rate Control (Whitney 72)

$$\delta x = J(\theta) \delta \theta$$

Outside singularities

$$\delta \theta = J^{-1}(\theta) \delta x$$

Arm at Configuration θ

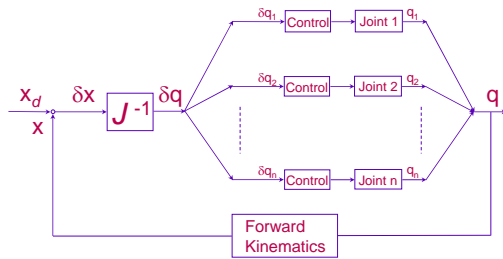
$$x = f(\theta)$$

$$\delta x = x_d - x$$

$$\delta \theta = J^{-1} \delta x$$

$\theta^+ = \theta + \delta \theta$

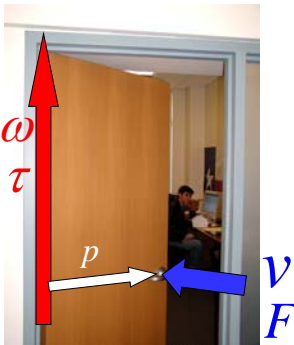
Resolved Motion Rate Control



Jacobian

- Differential Motion
- Linear & Angular Motion
- Velocity Propagation
- Explicit Form
- Static Forces

Angular/Linear – Velocities/Forces



$$v = \omega \times p$$

$$\tau = p \times F$$

Angular/Linear – Velocities/Forces

$$v = \omega \times p$$

$$v = -\hat{p} \omega$$

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} -p_y \\ p_x \end{pmatrix} \dot{\theta}$$

$$v = J \dot{\theta}$$

$$\tau = p \times F$$

$$\tau = \hat{p} F$$

$$\tau = (-\hat{p})^T F$$

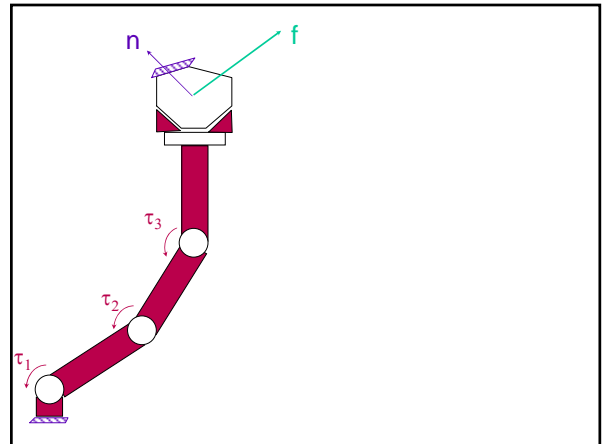
$$\tau = \begin{pmatrix} -p_y & p_x \end{pmatrix} \begin{pmatrix} F_x \\ F_y \end{pmatrix}$$

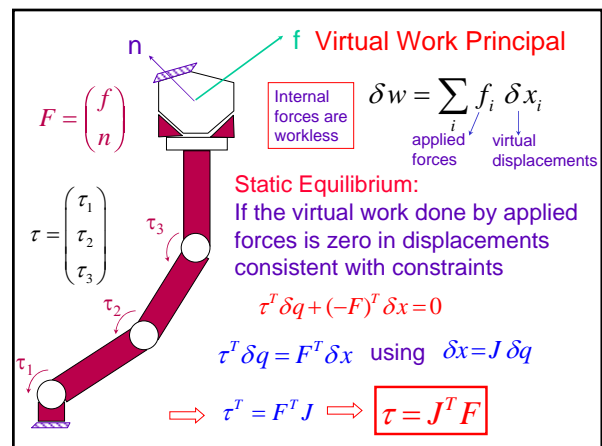
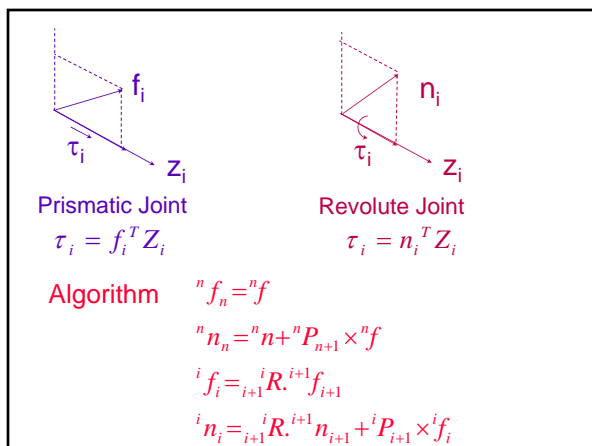
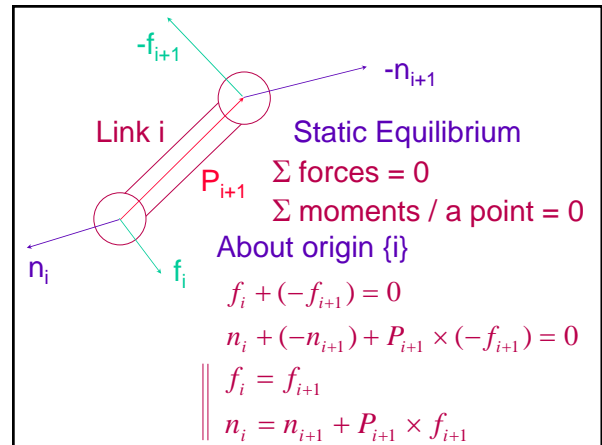
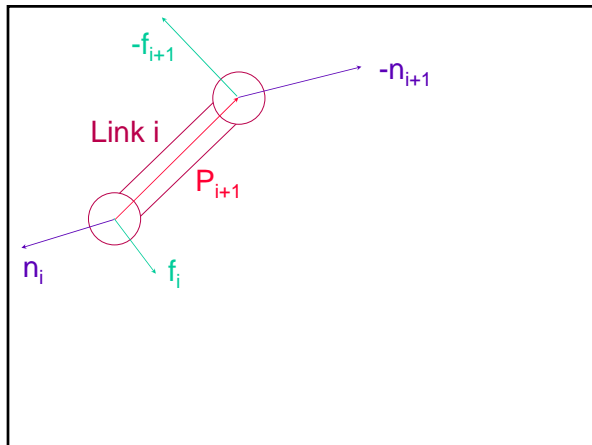
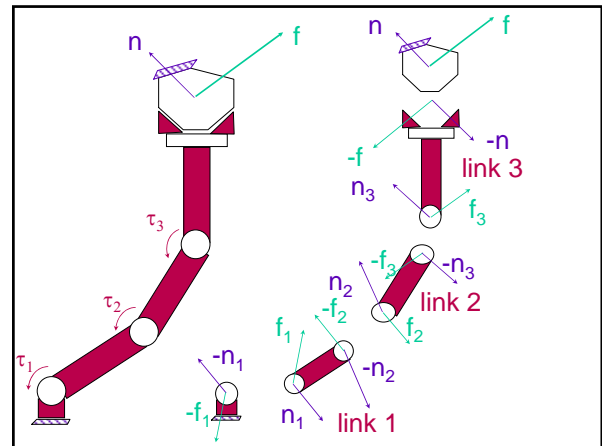
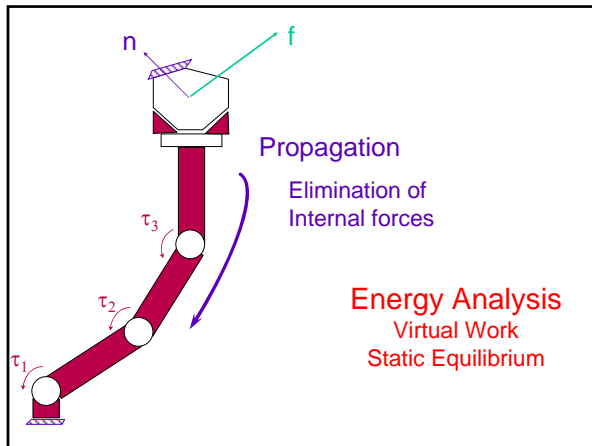
$$\tau = J^T F$$

Velocity/Force Duality

$$\dot{x} = J \dot{\theta}$$

$$\tau = J^T F$$



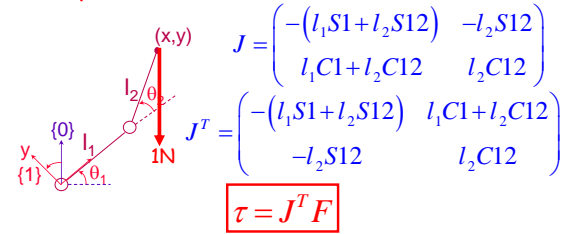


Velocity/Force Duality

$$\dot{x} = J \dot{\theta}$$

$$\tau = J^T F$$

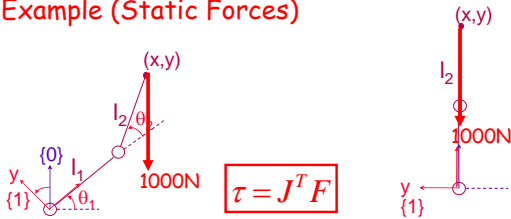
Example (Static Forces)



$$l_1 = l_2 = 1; \quad \theta_1 = 0; \quad \theta_2 = 60^\circ$$

$$\tau = \begin{pmatrix} -(l_1 S1 + l_2 S12) & l_1 C1 + l_2 C12 \\ -l_2 S12 & l_2 C12 \end{pmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = - \begin{bmatrix} l_1 C1 + l_2 C12 \\ l_2 C12 \end{bmatrix} = - \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}$$

Example (Static Forces)



$$\tau = \begin{pmatrix} -(l_1 S1 + l_2 S12) & l_1 C1 + l_2 C12 \\ -l_2 S12 & l_2 C12 \end{pmatrix} \begin{bmatrix} 0 \\ -1K \end{bmatrix} = \begin{bmatrix} l_1 C1 + l_2 C12 \\ l_2 C12 \end{bmatrix} (-1K) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$l_1 = l_2 = 1; \quad \theta_1 = 90; \quad \theta_2 = 0^\circ$$