Inverse Kinematics

Linearized Kinematic Model

$$\delta x = J(q)\,\delta q$$

Resolved Motion-Rate

(Whitney 1972)

$$\delta q = J^{-1}(q)\,\delta x$$

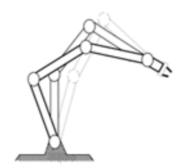
Jacobian

$$\begin{pmatrix}\Delta x \\ \Delta y\end{pmatrix} = J_{2\times 2}\begin{pmatrix}\Delta q_1 \\ \Delta q_2\end{pmatrix}$$

Inverse Jacobian

$$\begin{pmatrix}\Delta q_1 \\ \Delta q_2\end{pmatrix} = J^{-1}_{2\times 2}\begin{pmatrix}\Delta x \\ \Delta y\end{pmatrix}$$

Redundancy

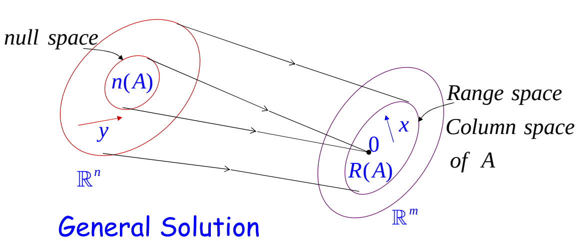


$$\begin{pmatrix}\Delta x \\ \Delta y\end{pmatrix} = J_{2\times 3}\begin{pmatrix}\Delta q_1 \\ \Delta q_2 \\ \Delta q_3\end{pmatrix}$$

Generalized Inverse

$$\begin{pmatrix}\Delta q_1 \\ \Delta q_2 \\ \Delta q_3\end{pmatrix} = J^{\#}_{3\times 2}\begin{pmatrix}\Delta x_1 \\ \Delta x_2\end{pmatrix} + \left[I - J^{\#}J\right]_{3\times 3}\begin{bmatrix}\Delta q_1 \\ \Delta q_2 \\ \Delta q_3\end{bmatrix}$$

System $\quad A_{(m\times n)}\,y_{(n\times 1)} = x_{(m\times 1)}$



General Solution

$$y = A^{\#}x + \left[I_n - A^{\#}A\right]y_0$$

Generalized Inverse

$A_{(m\times n)}$; $rank(A) = r$

$A^{\#}_{(n\times m)}$: $\quad AA^{\#}A = A$

Example

$$A = (2 \quad -1)$$

$$A^{\#} = \begin{pmatrix}\frac{1}{2} + \frac{a}{2} \\ a\end{pmatrix}$$

Example

$$Ay = \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = x$$

$$A^{\#} x = \begin{pmatrix} \frac{1}{2} + \frac{a}{2} \\ a \end{pmatrix} x = \begin{pmatrix} \frac{1}{2}(1+a)x \\ ax \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$Ay = \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{2}(1+a)x \\ ax \end{pmatrix} = x$$

$$A_{(m\times n)} y_{(n\times 1)} = x_{(m\times 1)}$$

$\dfrac{n > m}{(r = m)}$ → Less equations than unknowns

→ Free variables

→ ∞ solutions

Example $\begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

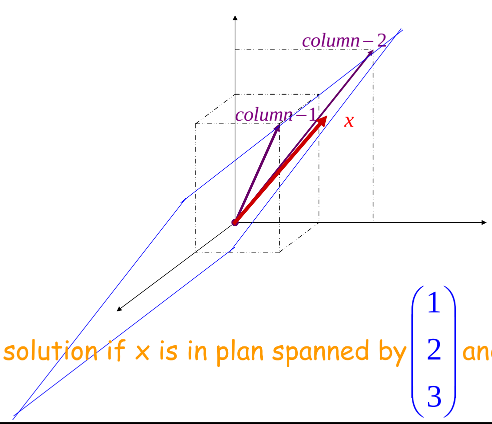
$\dfrac{n < m}{(r = n)}$ → More equations than unknowns

→ At most one solution

Example $\begin{pmatrix} 1 & 0 \\ 2 & 4 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$y_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + y_2 \begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

solution if x is in plan spanned by $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix}$

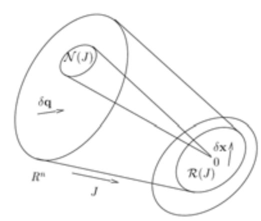


solution if x is in plan spanned by $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix}$

Jacobian Generalized Inverse

Generalized Inverse

$J_0^{\#}: \quad J_0 J_0^{\#} J_0 = J_0$

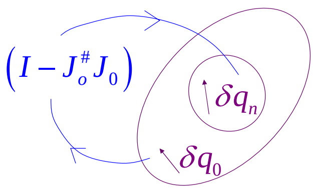


General Solution

$$\delta q = J_0^{\#}\, \delta x_0 + \left[I_n - J_0^{\#} J_0 \right] \delta q_0$$

General Solution

$$\delta q = J_0^{\#}\,\delta x_0 + \underbrace{\left[I_n - J_0^{\#} J_0\right]\delta q_0}_{\delta q_n}$$



$$\delta q_n = \left(I - J_0^{+} J_0\right)\delta q_0$$

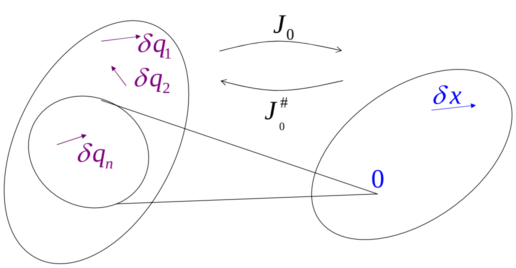
$$0 = J_0\,\delta q_n$$

$$\Rightarrow \quad 0 = J_0\left(I - J_0^{+} J_0\right)\delta q_0$$

$$0 \equiv J_0 - J_0 J_0^{+} J_0$$

$$\Rightarrow$$

$$J_0^{+}: \quad J_0 \triangleq J_0 J_0^{+} J_0$$



$$J_0^{\#}: J_0 = J_0 J_o^{\#} J_0$$

Pseudo Inverse

$$A A^{+} A = A$$

$$A^{+} A A^{+} = A^{+}$$

$$\left(A^{+} A\right)^{T} = A^{+} A$$

$$\left(A A^{+}\right)^{T} = A A^{+}$$

A^{+} : *unique*

Pseudo-Inverse

Left Inverse
$m > n$ $\quad A^{+} = \left(A^{T} A\right)^{-1} A^{T}$
$(r = n)$ $\quad A^{+} A = I$

$m = n = r$ $\quad A^{+} = A^{-1}$
$A^{+} A = A A^{+} = I$

Right Inverse
$m < n$ $\quad A^{+} = A^{T}(AA^{T})^{-1}$
$(r = m)$ $\quad A A^{+} = I$

Generalized Inverse

Left Inverse
$m > n$ $\quad A^{\#} = \left(A^{T} W^{-1} A\right)^{-1} A^{T} W^{-1}$
$(r = n)$ $\quad A^{\#} A = I$

$m = n = r$ $\quad A^{\#} = A^{-1}$
$A^{\#} A = A A^{\#} = I$

Right Inverse
$m < n$ $\quad A^{\#} = W^{-1} A^{T}(A W^{-1} A^{T})^{-1}$
$(r = m)$ $\quad A A^{\#} = I$

Instantaneous Inverse Kinematics

The solution, if it exists, of the system of m equations

$$\delta x_{(m\times 1)} = J_{(m\times n)}(q)\,\delta q_{(n\times 1)}$$

Definition-Theorem

The system: $\delta x = J(q)\,\delta q$ is said to be consistent and posses at least one solution if and only if

$$rank(J) = rank(J\,|\,\delta x)$$

The columns of J span $\mathbb{R}^m \rightarrow rank(J) = m$

solution for every δx

$$\delta x = J\delta q = \left(J_1 J_2 J_3 \cdots J_n\right)\begin{pmatrix}\delta q_1\\ \delta q_2\\ \vdots\\ \delta q_n\end{pmatrix}$$

Linear Combination of columns of J

$$\delta x = J_1\delta q_1 + J_2\delta q_2 + \cdots + J_n\delta q_n$$

Reduction to the Basic Kinematic Model

Initial Problem (m equations)

$$J\,\delta q = \delta x$$

Reduced Problem (m_0 equations) $J = E\,J_0$

$$\delta x = E(X)\,\delta x_0$$

$$J_0(q)\,\delta q = \delta x_0$$

Solving $\delta x = E(X)\,\delta x_0$

$E(X): m \times m_0$ $matrix$ $(m \geq m_0)$

$-rank\left(E(X)\right) \leq m_0$

$-rank\left(E(X)\right) < m_0$ at configuration x where the representation is singular

Left Inverse

If $rank\left(E(X)\right) = m_0$ the system has a unique solution:

$$\delta x_0 = E^{+}_{(m_0\times m)}(X)\;\delta x$$

E^{+}: is such that $E^{+}E = \mathrm{I}_{m_0}$

$$E^{+} = \left(E^{T}E\right)^{-1}E^{T}$$

and

$$E^{+}(X) = \begin{pmatrix} E^{+}_{p}(Xp) & 0 \\ 0 & E^{+}_{r}(Xr)\end{pmatrix}$$

System

$$\delta x_{m\times 1} = E_{m\times m_0}\,\delta x_{0\,m_0\times 1}$$

$$E^T_{m_0\times m}\,\delta x_{m\times 1} = \left(E^T E\right)_{m_0\times m_0}\,\delta x_{0\,m_0\times 1}$$

$$\left(E^T E\right)^{-1} E^T \delta x = \delta x_0$$

$$\delta x_0 = E^{\#}\delta x$$

$$E^+ = \left(E^T E\right)^{-1} E^T$$

$$E^+ E = \underbrace{\left(E^T E\right)^{-1} E^T}_{\text{Left Inverse}} E = I$$

Position Representations

Cartesian Coordinates (x, y, z)

$$E_P(X) = I_3$$

Cylindrical Coordinates (ρ, θ, z)

Using $(x \;\; y \;\; z)^T = (\rho\cos\theta \;\; \rho\sin\theta \;\; z)^T$

$$E_P(X) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta/\rho & \cos\theta/\rho & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Position Representations

Cartesian Coordinates (x, y, z)

$$E_P^+(X) = E_P^{-1}(X) = I_3$$

Cylindrical Coordinates (ρ, θ, z)

$$E_P^{-1}(X) = \begin{pmatrix} \cos\theta & -\rho\sin\theta & 0 \\ -\sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Spherical Coordinates (ρ, θ, ϕ)

Using

$(x \;\; y \;\; z)^T = (\rho\cos\theta\sin\phi \;\; \rho\sin\theta\sin\phi \;\; \rho\cos\theta)^T$

$$E_P(X) = \begin{pmatrix} \cos\theta\sin\phi & \sin\theta\sin\phi & \cos\phi \\ -\sin\theta/(\rho\sin\phi) & \cos\theta/(\rho\sin\phi) & 0 \\ \cos\theta\cos\phi/\rho & \sin\theta\cos\phi/\rho & -\sin\phi/\rho \end{pmatrix}$$

Spherical Coordinates (ρ, θ, ϕ)

$$E_P^{-1}(X) = \begin{pmatrix} \cos\theta\sin\phi & \rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ -\sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{pmatrix}$$

Rotation Representations

Direction Cosines

$$x_r = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}; \quad E_r(x_r) = \begin{pmatrix} -\hat{s}_1 \\ -\hat{s}_2 \\ -\hat{s}_3 \end{pmatrix}$$

$$E_r^+ = \left(E_r^T E_r\right)^{-1} E_r^T$$

$$\left(E_r^T E_r\right)^{-1} = \left(\hat{S}_1^T\hat{S}_1 + \hat{S}_2^T\hat{S}_2 + \hat{S}_3^T\hat{S}_3\right)^{-1}$$

Example

$$S = (S_1 S_2 S_3) = \begin{pmatrix} C1 & -S1 & 0 \\ S1 & C1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\hat{z} = \widehat{\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}} = \begin{pmatrix} 0 & -z_3 & z_2 \\ z_3 & 0 & -z_1 \\ -z_2 & z_1 & 0 \end{pmatrix}$$

$$\hat{S}_1 = \begin{bmatrix} 0 & 0 & S1 \\ 0 & 0 & -C1 \\ -S1 & C1 & 0 \end{bmatrix}$$

$$\hat{S}_1^T \hat{S}_1 = \begin{bmatrix} S_1^2 & -SC1 & 0 \\ -SC1 & C_1^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad SC1 = S1\,C1$$

$$\hat{S}_2 = \begin{bmatrix} 0 & 0 & C1 \\ 0 & 0 & S1 \\ -C1 & -S1 & 0 \end{bmatrix}$$

$$\hat{S}_2^T \hat{S}_2 = \begin{bmatrix} C_1^2 & SC1 & 0 \\ SC1 & S_1^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\hat{S}_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\hat{S}_3^T \hat{S}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E^T E = \sum \hat{S}_i^T \hat{S}_i = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 2I_3$$

$$\forall X_r = \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix} \Rightarrow E^T E = \sum \hat{S}_i^T \hat{S}_i = 2I_3$$

$$\left(E_r^T E_r\right)^{-1} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}^{-1} = \tfrac{1}{2} I_3$$

$$E_r^+ = \left(E_r^T E_r\right)^{-1} E_r^T = \frac{1}{2} E_r^T$$

$$E_r^T = \tfrac{1}{2}\left(-\hat{S}_1^T - \hat{S}_2^T - \hat{S}_3^T\right)$$

$$E_r^T = \tfrac{1}{2}\left(\hat{S}_1 \hat{S}_2 \hat{S}_3\right)$$

Angular Velocity

$$X_r = \begin{pmatrix} S1 \\ S2 \\ S3 \end{pmatrix}; \qquad \dot{X}_r = E_r \omega$$

Solution

$$\omega = \frac{1}{2} E^T \dot{X}_r$$

Direction Cosines - Rotation Error

Instantaneous Angular Error

$$x_r = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix}; \; x_{rd} = \begin{bmatrix} S_{1d} \\ S_{2d} \\ S_{3d} \end{bmatrix}$$

$$\delta x_r = \begin{pmatrix} S1 \\ S2 \\ S3 \end{pmatrix} - \begin{pmatrix} S1d \\ S2d \\ S3d \end{pmatrix}$$

$$\omega = \frac{1}{2} E^T \dot{X}_r$$

$$\delta\phi = \frac{1}{2} E^T \delta X_r$$

$$\delta X_r = \begin{pmatrix} S1 \\ S2 \\ S3 \end{pmatrix} - \begin{pmatrix} S1d \\ S2d \\ S3d \end{pmatrix}$$

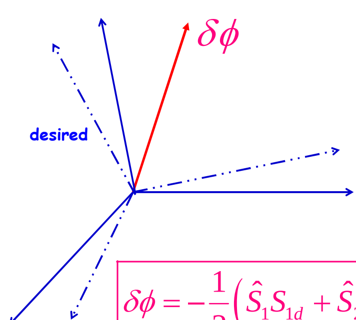
$$E_r^+ = \frac{1}{2} E^T$$

$$E^T = \left(-\hat{S}_1^T \; -\hat{S}_2^T \; -\hat{S}_3^T \right)$$

$$E_r^+ = \frac{1}{2} \left(\hat{S}_1 \; \hat{S}_2 \; \hat{S}_3 \right)$$

$$E_r^+ \begin{pmatrix} S1 \\ S2 \\ S3 \end{pmatrix} = \frac{1}{2} (\hat{S}_1 S1 + \hat{S}_2 S2 + \hat{S}_3 S3) \equiv 0$$

Instantaneous Angular Error



$$\delta\phi = -\frac{1}{2} \left(\hat{S}_1 S_{1d} + \hat{S}_2 S_{2d} + \hat{S}_3 S_{3d} \right)$$

Rotation Representations

Direction Cosines

Observing $E_r^T(x_r) E_r(x_r) = 2 I_3$

$$E_r^+(x_r) = \frac{1}{2} \left(-\hat{S}_1^T \; -\hat{S}_2^T \; -\hat{S}_3^T \right)$$

Euler Angles

$$E_r(X) = \begin{pmatrix} -S\varphi C\theta / S\theta & C\varphi C\theta / S\theta & 1 \\ C\varphi & S\varphi & 0 \\ S\varphi / S\theta & -C\varphi / S\theta & 0 \end{pmatrix}$$

$$E_r^{-1}(x_r) = \begin{pmatrix} 0 & \cos\psi & \sin\psi \sin\theta \\ 0 & \sin\psi & -\cos\psi \sin\theta \\ 1 & 0 & \cos\theta \end{pmatrix}$$

Euler Parameters

$$x_r = \lambda = (\lambda_0\ \lambda_1\ \lambda_2\ \lambda_3)^T$$

$$\dot{\lambda} = \frac{1}{2}\check{\lambda}\,\omega$$

$$\check{\lambda} = \begin{pmatrix} -\lambda_1 & -\lambda_2 & -\lambda_3 \\ \lambda_0 & \lambda_3 & -\lambda_2 \\ -\lambda_3 & \lambda_0 & \lambda_1 \\ \lambda_2 & -\lambda_1 & \lambda_0 \end{pmatrix}$$

$$E_r(\lambda) = \frac{1}{2}\check{\lambda}$$

Euler Parameters

Observing

$$\check{\lambda}^T \check{\lambda} = I_3$$

$$E_r^+(x_r) = 2\begin{pmatrix} -\lambda_1 & \lambda_0 & -\lambda_3 & \lambda_2 \\ -\lambda_2 & \lambda_3 & \lambda_0 & -\lambda_1 \\ -\lambda_3 & -\lambda_2 & \lambda_1 & \lambda_0 \end{pmatrix}$$

Inverse of the Basic Kinematic Model

System

$$\delta x_{0(m_0\times 1)} = J_0(q)_{(m_0\times n)}\,\delta q_{(n\times 1)}\ ;\ m_o \le n$$

Right Inverse

A solution iff $rank\ J_0 = m_0$

$\exists$ an $n \times m_0$ right inverse $\quad J_0^\# \,/\, J_0 J_0^\# = I_{m_0}$

System

$$\delta x_{0(m_0\times 1)} = J_0(q)_{(m_0\times n)}\,\delta q_{(n\times 1)}\ ;\ m_o \le n$$

Solution $\qquad \delta q = J_0^\#\,\delta X_0$

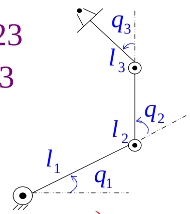
$J_0^\#$: Generalized Inverse

General Solution

$$\delta q = J_0^\#\,\delta X_0 + \underbrace{\left[I_n - J_0^\# J_0\right]\delta q_0}_{\delta q_n}$$

Redundancy (w.r.t a Task)

$$x = l_1 C1 + l_2 C12 + l_3 C123$$
$$y = l_1 S1 + l_2 S12 + l_3 S123$$



$$q_1 = q_2 = 0$$
$$l_1 = l_2 = l_3 = 1$$

$$J_{2\times 3}(q) = \begin{pmatrix} -S3 & -S3 & -S3 \\ 2+C3 & 1+C3 & C3 \end{pmatrix}$$

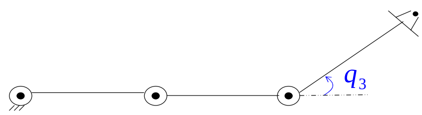
$$J^+_{3\times 2} = J^T \left(J\, J^T\right)^{-1}$$

$$J\, J^T = \begin{pmatrix} 3S_3^2 & -3(1+C3)\,S3 \\ -3(1+C_3)\,S3 & 3C_3^2 + 6C3 + 5 \end{pmatrix}$$

$$\det\left(J\, J^T\right) = 6S_3^2$$

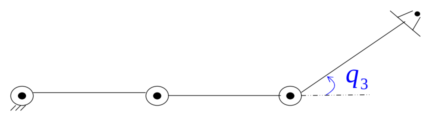
$$\left(J\, J^T\right)^{-1} = \frac{1}{6S_3^2}\begin{pmatrix} 3C_3^2 + 6C3 + 5 & 3(1+C3)\,S3 \\ 3(1+C_3)\,S3 & 3S_3^2 \end{pmatrix}$$

$$J^+_{(3\times 2)} = \frac{1}{6S3}\begin{pmatrix} 1+3C3 & 3S3 \\ -2 & 0 \\ -(5+3C3) & -3S3 \end{pmatrix}$$



$$\delta q = J^+\,\delta x + \left(I_n - J_0^+ J_0\right)\delta q_0$$

$$J^+ J = \frac{1}{6}\begin{pmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{pmatrix}$$



$$\delta q = J^+\,\delta x + \left(I_n - J_0^+ J_0\right)\delta q_0$$

$$\left(I - J^+\, J\right) = \frac{1}{6}\begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix}$$

$$\delta q_n = \frac{1}{6}\begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix}\delta q_0$$

$$\delta q_{n1} = \frac{1}{6}\left(\delta q_1 - 2\delta q_2 + \delta q_3\right)$$

$$\delta q_{n2} = \frac{1}{6}\left(-2\delta q_1 + 4\delta q_2 - 2\delta q_3\right)$$

$$\delta q_{n3} = \frac{1}{6}\left(\delta q_1 - 2\delta q_2 + \delta q_3\right)$$

Redundancy

System

$$\delta x_{0(m_0\times 1)} = J_0(q)_{\,(m_0\times n)}\,\delta q_{(n\times 1)}\;;\; m_o \le n$$

General Solution

$$\delta q = J_0^{\#}\,\delta X_0 + \underbrace{\left[I_n - J_0^{\#} J_0\right]\delta q_0}_{\delta q_n}$$


Kinematic Singularity

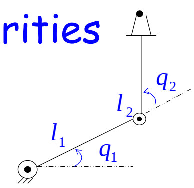
The Effector Locality loses the ability to move in a direction or to rotate about a direction - singular direction

$$J = \begin{pmatrix} J_1 & J_2 & \cdots & J_n \end{pmatrix}$$

$$\det(J) = 0$$

$$\det({}^i J) = \det({}^j J)$$

Kinematic Singularities



$$x = l_1 C1 + l_2 C12$$

$$y = l_1 S1 + l_2 S12$$

$$J = \begin{pmatrix} -(l_1 S1 + l_2 S12) & -l_2 S12 \\ l_1 C1 + l_2 S12 & l_2 C12 \end{pmatrix}$$

$$\det(J) = l_1 l_2 S2$$

Singularity at $q_2 = k\pi$

$$J = S_{01}\, J_{(1)}$$

$$J = \begin{pmatrix} C1 & -S1 \\ S1 & C1 \end{pmatrix} \begin{pmatrix} -l_2 S2 & -l_2 S2 \\ l_1 + l_2 C2 & l_2 C2 \end{pmatrix}$$

At Singularity

$$J = \begin{pmatrix} 0 & 0 \\ l_1 + l_2 & l_2 \end{pmatrix}$$

The rank of $(J\, J^T) < 2$

$(J^T J) < 2$

Singular Value Decomposition

Theorem - Definition

Any m x n matrix A of rank r can be factored into:

$A = U \Sigma V^T$; where

$-U$ is an m x m orthogonal matrix;

$-V$ is an n x n orthogonal matrix;

$-\Sigma$ is an m x n matrix of the form

$$\Sigma = \left(\begin{array}{c|c} \Sigma_r & 0 \\ \hline 0 & 0 \end{array}\right)$$; with

$\Sigma_r = diag[\sigma_i]$ with

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$$

$\sigma_i (i = 1, \cdots, r)$ are uniquely determined for A and called "Singular values of A"

Decomposition of A

$$A_{(m\times n)} = U_{(m\times m)} \Sigma_{(m\times n)} V^T{}_{(n\times n)}$$

$\underline{m \ge n}$ $\qquad A^T A = V\left(\Sigma^T \Sigma\right)V^T$

$$\Sigma^T \Sigma = \left(\begin{array}{c|c} \Sigma_r^2 & 0 \\ \hline 0 & 0 \end{array}\right)$$

$1^o)\ \det\left(A^T A - \sigma^2 I\right) = 0 \rightarrow \sigma_1^2, \cdots, \sigma_r^2 \rightarrow \Sigma$

$2^o)\left(A^T A - \sigma_i^2 I\right)V_i = 0 \rightarrow V$

$3^o)\ AV = U\Sigma$

$$\left(Av_1 \vdots Av_2 \vdots \cdots\right) = \left(\sigma_1 u_1 \vdots \sigma_2 u_2 \vdots \cdots\right)$$

$$u_i = \frac{Av_i}{\sigma_i} \rightarrow U$$

$\underline{m < n}$ $\qquad AA^T = U\left(\Sigma^T \Sigma\right)U^T$

$$\left(\Sigma\Sigma^T\right)_{(m\times m)} = \left(\begin{array}{c|c} \Sigma_r^2 & 0 \\ \hline 0 & 0 \end{array}\right)$$

$1^o)\ \det\left(AA^T - \sigma^2 I\right) = 0 \rightarrow \Sigma$

$2^o)\left(AA^T - \sigma_i^2 I\right)U_i = 0 \rightarrow U$

$3^o)\ A^T U = V\Sigma$

$$\left(A^T u_1 \vdots A^T u_1 \vdots \cdots\right) = \left(\sigma_1 v_1 \vdots \sigma_2 v_2 \vdots \cdots\right)$$

$$v_i = \frac{A^T u_i}{\sigma_i} \rightarrow V$$

Pseudo Inverse of $A = U\Sigma V^T$ **is**

$$\underline{\underline{A^+ = V\Sigma^+ U^T}}$$

$AA^+A = A$

$\left(A^+A\right)^T = A^+A$

$\left(AA^+\right)^T = AA^+$

Pseudo Inverse of $\Sigma = \begin{pmatrix} \sigma_i & & 0 \\ 0 & \sigma_r & \\ & & 0 \end{pmatrix}$ **is**

$$\Sigma^+ = \begin{pmatrix} \frac{1}{\sigma_i} & & 0 \\ 0 & \frac{1}{\sigma_i} & \\ & & 0 \\ & & \end{pmatrix}$$

Example

$$J_{(l)} = \begin{pmatrix} 0 & 0 \\ l_1 + l_2 & l_2 \end{pmatrix}$$

1^o) Σ?

$$J^T J = \begin{pmatrix} (l_1+l_2)^2 & (l_1+l_2)l_2 \\ (l_1+l_2)l_2 & l_2^2 \end{pmatrix}$$

$$\det\left(J^T J - \sigma^2 I\right) = 0$$

$$\sigma_1^2 = l_2^2 + (l_1 + l_1)^2$$

$$\sigma_2^2 = 0$$

$$\Sigma = \begin{pmatrix} \sqrt{l_2^2 + (l_1+l_2)^2} & 0 \\ 0 & 0 \end{pmatrix}$$

2^o) V?

$$\left(J^T J - \sigma_1^2 I\right) v_1 = 0$$

$$\begin{pmatrix} -l_2^2 & (l_1+l_2)l_2 \\ (l_1+l_2)l_2 & -(l_1+l_2)^2 \end{pmatrix} v_1 = 0$$

$$V = \frac{1}{\sqrt{l_2^2 + (l_1+l_2)}} \begin{pmatrix} l_1+l_2 & -l_2 \\ l_2 & l_1+l_2 \end{pmatrix}$$

3^o) U?

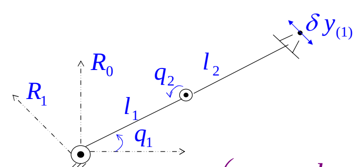
$$u_i = \frac{J v_i}{\sigma_i}$$

$$u_1 = \begin{pmatrix} 0 & 0 \\ \frac{l_1+l_2}{\sqrt{(.)}} & \frac{l_2}{\sqrt{(.)}} \end{pmatrix} \begin{pmatrix} \frac{l_1+l_2}{\sqrt{(.)}} \\ \frac{l_2}{\sqrt{(.)}} \end{pmatrix}$$

$$u_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

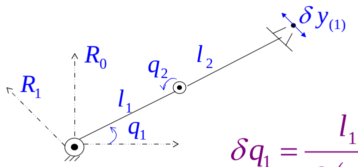
$$J = S_{01}\ U\ \Sigma\ V^T$$



$$\delta x_{(1)} = J_{(1)} \delta q$$

$$\delta q = J_{(1)}^{+}\ \delta x_{(1)}$$

$$\delta q = \begin{pmatrix} 0 & \frac{l_1+l_2}{l_2^2(l_1+l_2)^2} \\ 0 & \frac{l_2}{l_2^2(l_1+l_2)^2} \end{pmatrix} \delta x$$



$$\delta q_1 = \frac{l_1 + l_2}{l_2^2\left(l_1 + l_2\right)^2}\ \delta y_{(1)}$$

$$\delta q_2 = \frac{l_2}{l_2^2\left(l_1 + l_2\right)^2}\ \delta y_{(1)}$$

$$J = \begin{pmatrix} C1 & -S1 \\ S1 & C1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{l_2^2 + \left(l_1 + l_2\right)^2} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{l_1 + l_2}{\sqrt{(.)}} & \frac{l_2}{\sqrt{(.)}} \\ \frac{-l_2}{\sqrt{(.)}} & \frac{l_1 + l_2}{\sqrt{(.)}} \end{pmatrix}$$

$$J^+ = V\ \Sigma^+\ U^T\ S_{01}^T$$

$$J^+ = \begin{pmatrix} 0 & \frac{l_1 + l_2}{l_2^2 + \left(l_1 + l_2\right)^2} \\ 0 & \frac{l_2}{l_2^2 + \left(l_1 + l_2\right)^2} \end{pmatrix}\begin{pmatrix} C1 & S1 \\ -S1 & C1 \end{pmatrix}$$

General Solution

$$\delta q = J^+\ \delta x + \underbrace{\left(I - J^+ J\right)\ \delta q_0}_{\delta q_n}$$

$$J^+ J = \frac{1}{\sqrt{l_2^2 + \left(l_1 + l_2\right)}}\begin{pmatrix} l_1 + l_2 & -l_2 \\ l_2 & l_1 + l_2 \end{pmatrix}$$

$$\delta q_n = \frac{1}{l_2^2 + \left(l_1 + l_2\right)^2}\begin{pmatrix} l_2^2 & -\left(l_1 + l_2\right) l_2 \\ -\left(l_1 + l_2\right) l_2 & \left(l_1 + l_2\right)^2 \end{pmatrix}\delta q_0$$

$$\delta x_n = J\ \delta q_n = 0$$

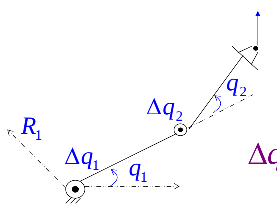
Problem with the Pseudo Inverse Solution

$$J = \begin{pmatrix} C1 & -S1 \\ S1 & C1 \end{pmatrix}\overbrace{\begin{pmatrix} -l_2 S2 & -l_2 S2 \\ l_1 + l_2 C2 & l_2 C2 \end{pmatrix}}^{J_{(1)}}$$

$$J^{-1} = \underbrace{\frac{1}{l_1 l_2 S2}\begin{pmatrix} l_2 C2 & l_2 S2 \\ -\left(l_1 + l_2 C2\right) & -l_2 S2 \end{pmatrix}}_{J_{(1)}^{-1}}\begin{pmatrix} C1 & S1 \\ -S1 & C1 \end{pmatrix}$$

small θ_2

$$J_{(1)}^{-1} \cong \begin{pmatrix} \frac{1}{l_1 \theta_2} & \frac{1}{l_1} \\ -\frac{l_1 + l_2}{l_1 l_2 \theta_2} & -\frac{1}{l_1} \end{pmatrix}$$

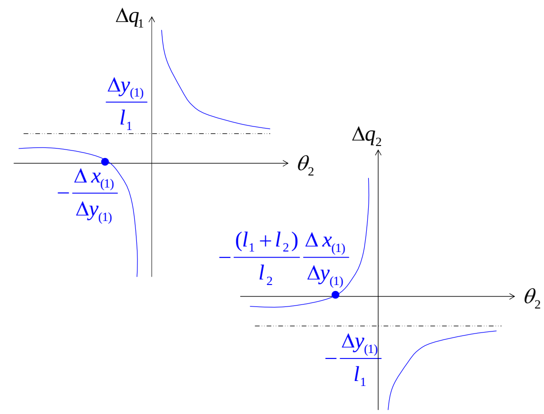


$$\Delta X = \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

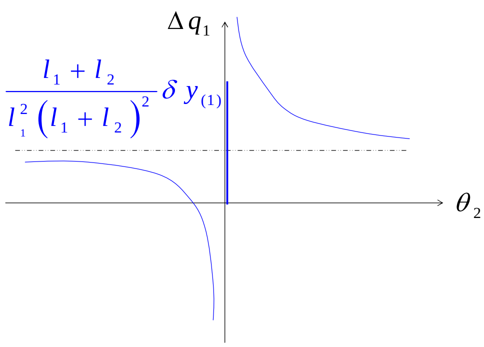
$$\Delta q = J^{-1} \Delta X$$

$$\Delta q_1 = \frac{\Delta x_{(1)}}{l_1} \cdot \frac{1}{\theta_2} + \frac{\Delta y_{(1)}}{l_1}$$

$$\Delta q_2 = \frac{(l_1 + l_2)\Delta x_{(1)}}{l_1 l_2} \cdot \frac{1}{\theta_2} + \frac{\Delta y_{(1)}}{l_1}$$



Pseudo Inverse Solution



Singularity Robust Inverse

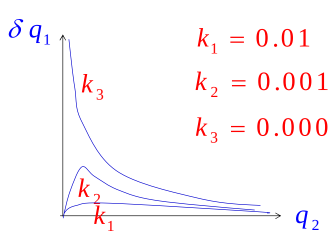
Pseudo-Inverse

$$J^{+} = J^{T} \left(J J^{T} \right)^{-1}$$

S-R Inverse

$$J^{*} = J^{T} \left(J J^{T} + k I \right)^{-1}$$

Singularity Robust Inverse



$k_1 = 0.01$

$k_2 = 0.001$

$k_3 = 0.0001$