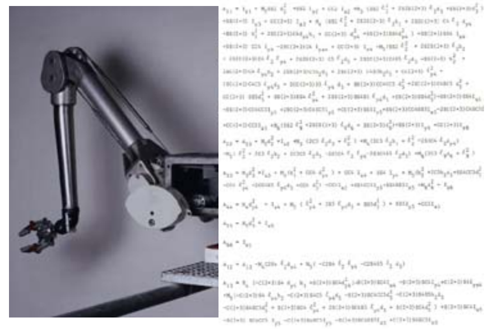
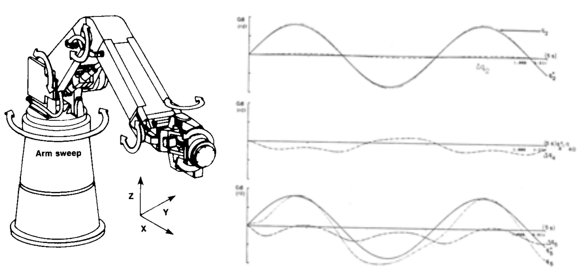
Dynamics



MA23



Joint Space Dynamics

$$A(q)\ddot{q} + b(q,\dot{q}) + g(q) = \Gamma$$

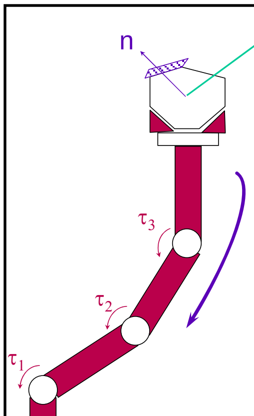
q: Joint Coordinates

$A(q)$: Kinetic Energy Matrix

$b(q,\dot{q})$: Centrifugal and Coriolis forces

$g(q)$: Gravity forces

Γ: Generalized forces



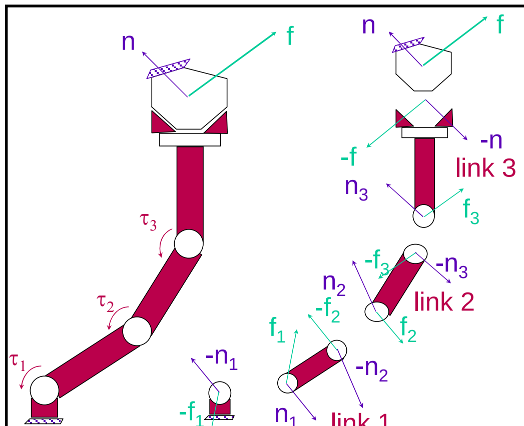
Propagation

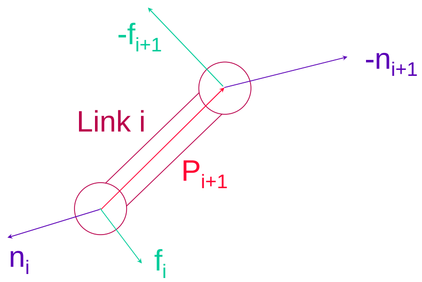
Elimination of Internal forces

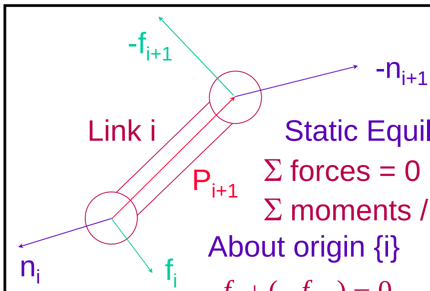
Energy Analysis

Virtual Work

Static Equilibrium







Static Equilibrium

Σ forces = 0

Σ moments / a point = 0

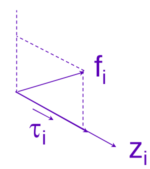
About origin {i}

$$f_i + (-f_{i+1}) = 0$$

$$n_i + (-n_{i+1}) + P_{i+1} \times (-f_{i+1}) = 0$$

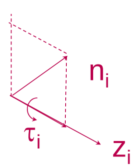
$$f_i = f_{i+1}$$

$$n_i = n_{i+1} + P_{i+1} \times f_{i+1}$$



Prismatic Joint

$$\tau_i = f_i^T Z_i$$



Revolute Joint

$$\tau_i = n_i^T Z_i$$

Algorithm

$$^n f_n = {}^n f$$

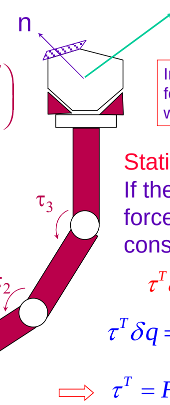
$$^n n_n = {}^n n + {}^n P_{n+1} \times {}^n f$$

$$^i f_i = {}_{i+1}^{\ i}R . {}^{i+1} f_{i+1}$$

$$^i n_i = {}_{i+1}^{\ i}R . {}^{i+1} n_{i+1} + {}^i P_{i+1} \times {}^i f_i$$

Virtual Work Principal

$$F = \begin{pmatrix} f \\ n \end{pmatrix}$$



$$\tau = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix}$$

Internal forces are workless

$$\delta w = \sum_i f_i \, \delta x_i$$

applied forces; virtual displacements

Static Equilibrium:
If the virtual work done by applied forces is zero in displacements consistent with constraints

$$\tau^T \delta q + (-F)^T \delta x = 0$$

$$\tau^T \delta q = F^T \delta x \quad \text{using} \quad \delta x = J\,\delta q$$

$$\Rightarrow \tau^T = F^T J \Rightarrow \boxed{\tau = J^T F}$$

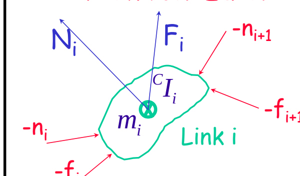
Velocity/Force Duality

$$\boxed{\dot{x} = J\,\dot{\theta}}$$

$$\boxed{\tau = J^T F}$$

Formulations

Newton-Euler



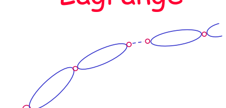
Newton: $F_i = m_i \dot{\mathbf{v}}_{C_i}$

Euler: $N_i = {}^{C_i}I \dot{\omega}_i + \omega_i \times {}^{C_i}I \omega_i$

Eliminate Internal Forces and Moments

$$\Gamma_i = \begin{cases} n_i^T . Z_i & \text{revolute} \\ f_i^T . Z_i & \text{prismatic} \end{cases}$$

Lagrange



Kinetic Energy: $\sum_i T_i$

Potential Energy

Generalized Coordinates

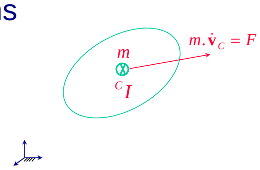
$$T = \frac{1}{2} \dot{q}^T A \dot{q}$$

$$A\ddot{q} + b + g = \Gamma$$

Newton-Euler Equations

Translational Motion

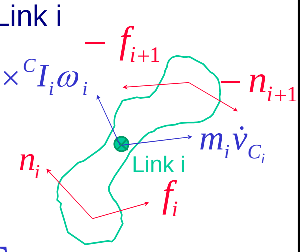
$$m\dot{\mathbf{v}}_C = F$$



Rotational Motion

$$^C I\dot{\omega} + \omega \times {}^C I\omega = N$$

Dynamic forces on Link i



$$^C I_i\dot{\omega}_i + \omega_i \times {}^C I_i\omega_i$$

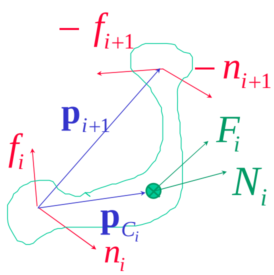
$$m_i\dot{\mathbf{v}}_{C_i} = \sum forces$$

$$^C I_i\dot{\omega}_i + \omega_i \times {}^C I_i\omega_i = \sum moments / c_i$$

Inertial forces/moments

$$F_i = m_i\dot{\mathbf{v}}_{C_i}$$

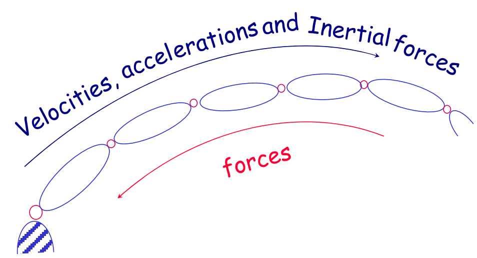
$$N_i = {}^C I_i\dot{\omega}_i + \omega_i \times {}^C I_i\omega_i$$



$$F_i = f_i - f_{i+1}$$

$$N_i = n_i - n_{i+1} + (-\mathbf{p}_{C_i}) \times f_i + (\mathbf{p}_{i+1} - \mathbf{p}_{C_i}) \times (-f_{i+1})$$

Newton-Euler Algorithm



Lagrange Equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \tau$$

Lgrangian → $L = T - U$ (Kinetic Energy: T; Potential Energy: U)

Since $U = U(q)$

$$\Rightarrow \underbrace{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q}}_{\text{Inertial forces}} + \frac{\partial U}{\partial q} = \tau$$

Gravity vector: $\frac{\partial U}{\partial q}$

Lagrange Equations

$$\underbrace{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q}}_{\text{Inertial forces}} = \Gamma - g; \quad g = \frac{\partial U}{\partial q}$$

$$\Downarrow$$

$$A(q)\ddot{q} + b(q, \dot{q}) = \Gamma - g(q)$$

Inertial forces

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q}\right) - \frac{\partial T}{\partial q} = \Gamma - g \qquad T = \frac{1}{2}\dot q^T A(q)\dot q$$

$$\frac{\partial T}{\partial \dot q} = \frac{\partial}{\partial \dot q}\left[\frac{1}{2}\dot q^T A(q)\,\dot q\right] \quad = A(q)\dot q$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q}\right) = \frac{d}{dt}(A\dot q) = A\ddot q + \dot A\dot q$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q}\right) - \frac{\partial T}{\partial q} = A\ddot q + \dot A\dot q - \frac{1}{2}\begin{bmatrix} \dot q^T \frac{\partial A}{\partial q_1}\dot q \\ \vdots \\ \dot q^T \frac{\partial A}{\partial q_n}\dot q \end{bmatrix} = A\ddot q + b(q,\dot q)$$

$$* \; \frac{\partial T}{\partial \dot q} = A\dot q \qquad \left[T = \frac{1}{2}m\dot x^2;\; \frac{\partial}{\partial \dot x}\left(\frac{1}{2}m\dot x^2\right) = \blacksquare\right]$$

$$T = \frac{1}{2}\dot q^T A(q)\dot q;\; v = A^{1/2}\dot q \rightarrow T = \frac{1}{2}v^T v$$

$$\frac{\partial T}{\partial \dot q} = \frac{\partial T}{\partial v}\frac{\partial v}{\partial \dot q} \quad = A^{1/2}v = A\dot q$$

$$\frac{\partial}{\partial \mathbf{v}}\left(\frac{1}{2}\mathbf{v}^T\mathbf{v}\right) = \mathbf{v} \qquad A^{1/2}$$

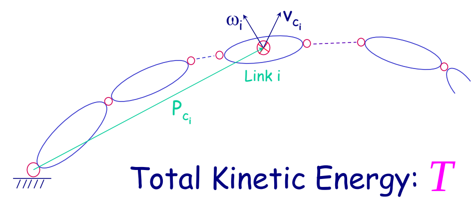
Equations of Motion

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q}\right) - \frac{\partial T}{\partial q} = A\ddot q + \dot A\dot q - \frac{1}{2}\begin{bmatrix} \dot q^T \frac{\partial A}{\partial q_1}\dot q \\ \vdots \\ \dot q^T \frac{\partial A}{\partial q_n}\dot q \end{bmatrix} = A\ddot q + b(q,\dot q)$$

$$\boxed{A(q)\ddot q + b(q,\dot q) + g(q) = \Gamma}$$

$$A(q): T = \frac{1}{2}\dot q^T A\,\dot q \qquad A(q) \;\Rightarrow\; b(q,\dot q)$$

Equations of Motion

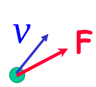


Total Kinetic Energy: T

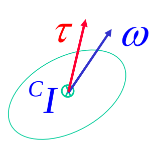
$$T = \sum T_{Link\,i} \equiv \frac{1}{2}\dot q^T A\,\dot q$$

Kinetic Energy

Work done by external forces to bring the system from rest to its current state.

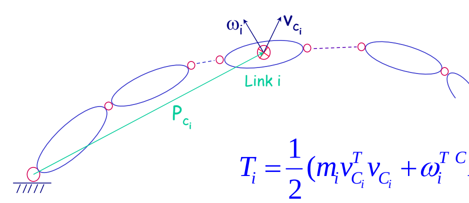


$$T = \frac{1}{2}mv^2$$



$$T = \frac{1}{2}\omega^T\, {}^C I\, \omega$$

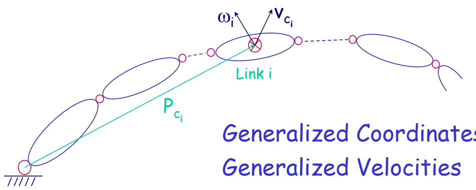
Equations of Motion — Explicit Form



$$T_i = \frac{1}{2}\left(m_i v_{C_i}^T v_{C_i} + \omega_i^T\, {}^C I_i \omega_i\right)$$

Total Kinetic Energy ⇨ $T = \sum_{i=1}^{n} T_i$

Equations of Motion — Explicit Form



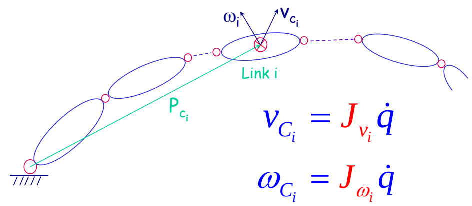
Generalized Coordinates q

Generalized Velocities $\dot{q}$

Kinetic Energy
Quadratic Form of
Generalized Velocities $\quad T = \frac{1}{2}\dot{q}^T A\,\dot{q}$

$$\frac{1}{2}\dot{q}^T A\,\dot{q} \equiv \frac{1}{2}\sum_{i=1}^{n}(m_i v_{C_i}^T v_{C_i} + \omega_i^T\,{}^C I_i \omega_i)$$

Equations of Motion — Explicit Form



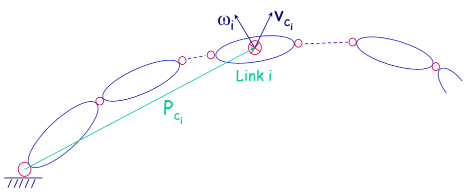
$$v_{C_i} = J_{v_i}\dot{q}$$

$$\omega_{C_i} = J_{\omega_i}\dot{q}$$

$$\frac{1}{2}\dot{q}^T A\,\dot{q} = \frac{1}{2}\sum_{i=1}^{n}(m_i v_{C_i}^T v_{C_i} + \omega_i^T\,{}^C I_i \omega_i)$$

$$= \frac{1}{2}\sum_{i=1}^{n}(m_i \dot{q}^T J_{v_i}^T J_{v_i}\dot{q} + \dot{q}^T J_{\omega_i}^T\,{}^C I_i J_{\omega_i}\dot{q})$$

Equations of Motion — Explicit Form

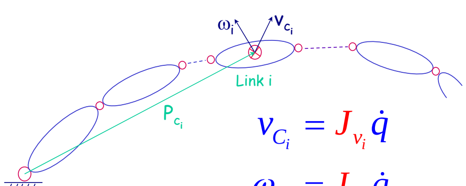


$$\frac{1}{2}\dot{q}^T A\,\dot{q} = \frac{1}{2}\dot{q}^T\left[\sum_{i=1}^{n}(m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T\,{}^C I_i J_{\omega_i})\right]\dot{q}$$

$$A = \sum_{i=1}^{n}(m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T\,{}^C I_i J_{\omega_i})$$

$$\underset{(n\times n)}{A(q)} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Equations of Motion — Explicit Form



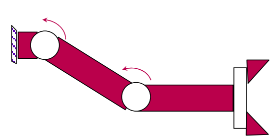
$$v_{C_i} = J_{v_i}\dot{q}$$

$$\omega_{C_i} = J_{\omega_i}\dot{q}$$

$$J_{v_i} = \left[\frac{\partial p_{C_i}}{\partial q_1} \quad \frac{\partial p_{C_i}}{\partial q_2} \quad \cdots \quad \frac{\partial p_{C_i}}{\partial q_i} \quad 0 \quad 0 \quad \cdots \quad 0\right]$$

$$J_{\omega_i} = \left[\bar{\varepsilon}_1 z_1 \quad \bar{\varepsilon}_2 z_2 \quad \cdots \quad \bar{\varepsilon}_i z_i \quad 0 \quad 0 \quad \cdots \quad 0\right]$$

Vector $b(q,\dot{q})$ — Centrifugal & Coriolis Forces



$$\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix}$$

Vector $b(q,\dot q)$

$\frac{\partial A}{\partial q_1}$ $\qquad$ $\frac{\partial a_{12}}{\partial q_1}$

$$b(q,\dot q)=\dot A\dot q-\frac{1}{2}\begin{bmatrix}\dot q^T A_{q_1}\dot q\\ \dot q^T A_{q_2}\dot q\end{bmatrix}=\begin{pmatrix}\dot a_{11} & \dot a_{12}\\ \dot a_{12} & \dot a_{22}\end{pmatrix}\dot q-\frac{1}{2}\begin{bmatrix}\dot q^T\begin{pmatrix}a_{111} & a_{121}\\ a_{121} & a_{221}\end{pmatrix}\dot q\\ \dot q^T\begin{pmatrix}a_{112} & a_{122}\\ a_{122} & a_{222}\end{pmatrix}\dot q\end{bmatrix}$$

$$\dot a_{12}=a_{111}\dot q_1+a_{112}\dot q_2$$

$$b(q,\dot q)=\begin{bmatrix}\frac{1}{2}(a_{111}+a_{111}-a_{111}) & \frac{1}{2}(a_{122}+a_{122}-a_{221})\\ \frac{1}{2}(a_{211}+a_{221}-a_{112}) & \frac{1}{2}(a_{222}+a_{222}-a_{222})\end{bmatrix}\begin{bmatrix}\dot q_1^2\\ \dot q_2^2\end{bmatrix}+\begin{bmatrix}a_{112}+a_{121}-a_{121}\\ a_{212}+a_{221}-a_{122}\end{bmatrix}[\dot q_1\dot q_2]$$

Christoffel Symbols

$$b_{ijk}=\frac{1}{2}(a_{ijk}+a_{ikj}-a_{jki})\qquad \frac{\partial a_{ij}}{\partial q_k}$$

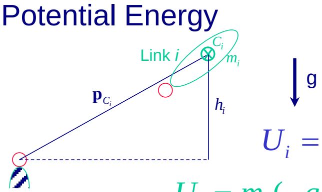
$$b(q,\dot q)=\begin{bmatrix}b_{111} & b_{122}\\ b_{211} & b_{222}\end{bmatrix}\begin{bmatrix}\dot q_1^2\\ \dot q_2^2\end{bmatrix}+\begin{bmatrix}2b_{112}\\ 2b_{212}\end{bmatrix}[\dot q_1\dot q_2]$$

$C(\mathbf{q})$ $\qquad$ $B(\mathbf{q})$

$$\underset{(n\times n)}{C(\mathbf{q})}\underset{(n\times 1)}{[\dot{\mathbf{q}}^2]}=\begin{bmatrix}b_{1,11} & b_{1,22} & \cdots & b_{1,nn}\\ b_{2,11} & b_{2,22} & \cdots & b_{2,nn}\\ \vdots & \vdots & \vdots & \vdots\\ b_{n,11} & b_{n,22} & \cdots & b_{n,nn}\end{bmatrix}\begin{bmatrix}\dot q_1^2\\ \dot q_2^2\\ \vdots\\ \dot q_n^2\end{bmatrix}$$

$$\underset{(n\times\frac{(n-1)n}{2})}{B(\mathbf{q})}\ \underset{(\frac{(n-1)n}{2}\times 1)}{[\dot{\mathbf{q}}\dot{\mathbf{q}}]}=\begin{bmatrix}2b_{1,12} & 2b_{1,13} & \cdots & 2b_{1,(n-1)n}\\ 2b_{2,12} & 2b_{2,13} & \cdots & 2b_{2,(n-1)n}\\ \vdots & \vdots & \vdots & \vdots\\ 2b_{n,12} & 2b_{n,13} & \cdots & 2b_{n,(n-1)n}\end{bmatrix}\begin{bmatrix}\dot q_1\dot q_2\\ \dot q_1\dot q_3\\ \vdots\\ \dot q_{(n-1)}\dot q_n\end{bmatrix}$$

Potential Energy



$$U_i=m_i g_0 h_i+U_0$$

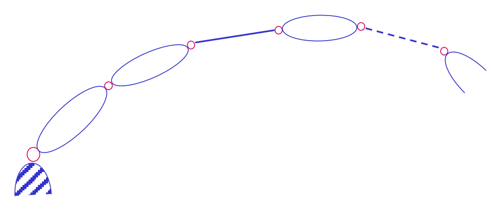
$$U_i=m_i(-g^T\mathbf{p}_{C_i});\ U=\sum_i U_i$$

Gravity Vector

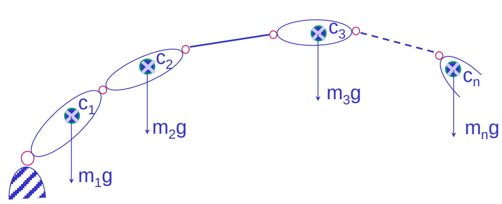
$$g_j=\frac{\partial U}{\partial q_j}=-\sum_{i=1}^{n}(m_i g^T\frac{\partial \mathbf{p}_{C_i}}{\partial q_j})$$

$$g=-\begin{pmatrix}J_{v_1}^T & J_{v_2}^T & \cdots & J_{v_n}^T\end{pmatrix}\begin{pmatrix}m_1 g\\ m_2 g\\ \vdots\\ m_n g\end{pmatrix}$$

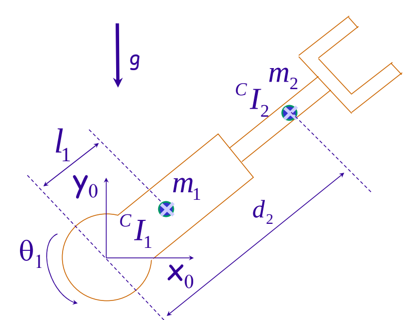
Gravity Vector



Gravity Vector



$$g=-(J_{v_1}^T(m_1 g)+J_{v_2}^T(m_2 g)+\cdots+J_{v_n}^T(m_n g))$$



Matrix A

$$A = m_1 J_{v_1}^T J_{v_1} + J_{\omega_1}^T \, {}^C I_1 J_{\omega_1} + m_2 J_{v_2}^T J_{v_2} + J_{\omega_2}^T \, {}^C I_2 J_{\omega_2}$$

J_{v_1} and J_{v_2} : direct differentiation of the vectors:

$${}^0\mathbf{p}_{C_1} = \begin{bmatrix} l_1 c_1 \\ l_1 s_1 \\ 0 \end{bmatrix}; \text{ and } {}^0\mathbf{p}_{C_2} = \begin{bmatrix} d_2 c_1 \\ d_2 s_1 \\ 0 \end{bmatrix}$$

In frame {0}, these matrices are:

$${}^0J_{v_1} = \begin{bmatrix} -l_1 c_1 & 0 \\ l_1 c_1 & 0 \\ 0 & 0 \end{bmatrix}; \text{ and } {}^0J_{v_2} = \begin{bmatrix} -d_2 s_1 & c_1 \\ d_2 c_1 & s_1 \\ 0 & 0 \end{bmatrix}$$

This yields

$$m_1({}^0J_{v_1}^T \, {}^0J_{v_1}) = \begin{bmatrix} m_1 l_1^2 & 0 \\ 0 & 0 \end{bmatrix}; \text{ and } m_2({}^0J_{v_2}^T \, {}^0J_{v_2}) = \begin{bmatrix} m_2 d_2^2 & 0 \\ 0 & m_2 \end{bmatrix}$$

The matrices J_{ω_1} and J_{ω_2} are given by

$$J_{\omega_1} = [\bar{\epsilon}_1 \mathbf{z}_1 \quad 0] = \text{ and } J_{\omega_2} = [\bar{\epsilon}_1 \mathbf{z}_1 \quad \bar{\epsilon}_2 \mathbf{z}_2]$$

Joint 1 is revolute and joint 2 is prismatic:

$${}^0J_{\omega_1} = {}^0J_{\omega_2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

And

$$({}^0J_{\omega_1}^T \, {}^C I_1 \, {}^0J_{\omega_1}) = \begin{bmatrix} I_{zz1} & 0 \\ 0 & 0 \end{bmatrix}; \text{ and } ({}^0J_{\omega_2}^T \, {}^C I_2 \, {}^0J_{\omega_2}) = \begin{bmatrix} I_{zz2} & 0 \\ 0 & 0 \end{bmatrix}$$

Finally,

$$A = \begin{bmatrix} m_1 l_1^2 + I_{zz1} + m_2 d_2^2 + I_{zz2} & 0 \\ 0 & m_2 \end{bmatrix}$$

Centrifugal and Coriolis Vector b

$$b_{i,jk} = \frac{1}{2}\left(a_{ijk} + a_{ikj} - a_{jki}\right)$$

where $a_{ijk} = \frac{\partial a_{ij}}{\partial q_k}$; with $b_{iii} = 0$ and $b_{iji} = 0$ for $i > j$

For this manipulator, only m_{11} is configuration dependent - function of d_2. This implies that only m_{112} is non-zero,

$$m_{112} = 2 m_2 d_2.$$

Matrix B $B = \begin{bmatrix} 2b_{112} \\ 0 \end{bmatrix} = \begin{bmatrix} 2m_2 d_2 \\ 0 \end{bmatrix}.$

Matrix C $C = \begin{bmatrix} 0 & b_{122} \\ b_{211} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -m_2 d_2 & 0 \end{bmatrix}.$

Vector b $b = \begin{bmatrix} 2m_2 d_2 \\ 0 \end{bmatrix} [\dot{\theta}_1 \dot{d}_2] + \begin{bmatrix} 0 & 0 \\ -m_2 d_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{d}_2^2 \end{bmatrix}.$

Vector b

$$b = \begin{bmatrix} 2m_2 d_2 \\ 0 \end{bmatrix} [\dot{\theta}_1 \dot{d}_2] + \begin{bmatrix} 0 & 0 \\ -m_2 d_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{d}_2^2 \end{bmatrix}.$$

The Gravity Vector g

$$g = -\left[J_{v_1}^T m_1 \mathbf{g} + J_{v_2}^T m_2 \mathbf{g} \right]$$

In frame {0}, ${}^0\mathbf{g} = (0 \quad -g \quad 0)^T$ and the gravity vector is

$${}^0g = -\begin{bmatrix} -l_1 s_1 & l_1 c_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -m_1 g \\ 0 \end{bmatrix} - \begin{bmatrix} -d_2 s_1 & d_2 c_1 & 0 \\ c_1 & s_1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -m_2 g \\ 0 \end{bmatrix}$$

and

$${}^0g = \begin{bmatrix} (m_1 l_1 + m_2 d_2) g c_1 \\ m_2 g s_1 \end{bmatrix}$$

Equations of Motion

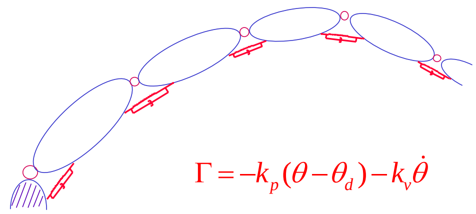
$$A(q)\ddot{q} + b(q, \dot{q}) + g(q) = \Gamma$$

$$\begin{bmatrix} m_1 l_1^2 + I_{zz1} + m_2 d_2^2 + I_{zz2} & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{d}_2 \end{bmatrix} + \begin{bmatrix} 2m_2 d_2 \\ 0 \end{bmatrix} [\dot{\theta}_1 \dot{d}_2] + \begin{bmatrix} 0 & 0 \\ -m_2 d_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{d}_2^2 \end{bmatrix} + \begin{bmatrix} (m_1 l_1 + m_2 d_2) g c_1 \\ m_2 g s_1 \end{bmatrix} = \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix}$$

Joint Space

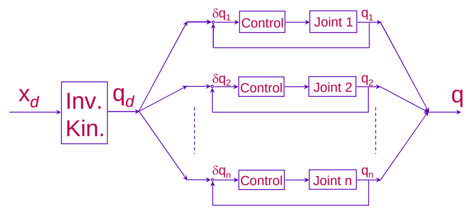
Control

Joint-Space Control



$$\Gamma = -k_p(\theta - \theta_d) - k_v\dot{\theta}$$

Joint Space Control



Resolved Motion Rate Control (Whitney 72)

$$\delta x = J(\theta)\delta\theta$$

Outside singularities

$$\delta\theta = J^{-1}(\theta)\delta x$$

Arm at Configuration θ

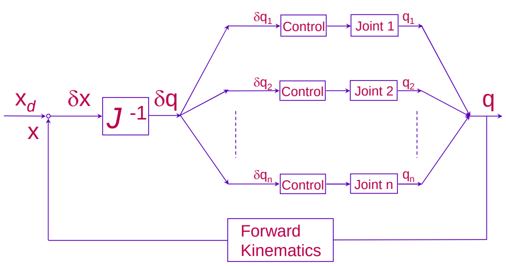
$$x = f(\theta)$$

$$\delta x = x_d - x$$

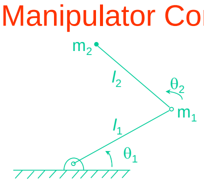
$$\delta\theta = J^{-1}\delta x$$

$$\boxed{\theta^{+} = \theta + \delta\theta}$$

Resolved Motion Rate Control



Manipulator Control

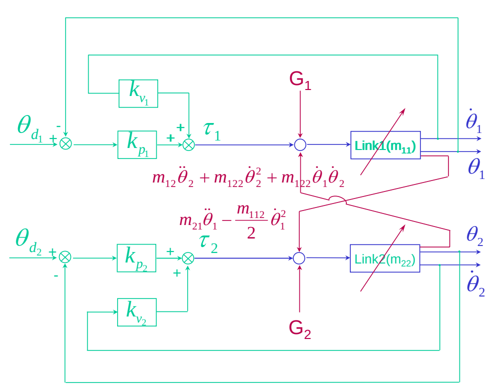


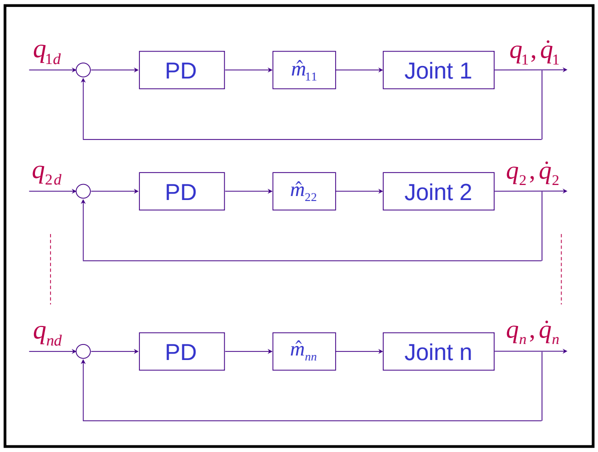
$$M(\theta)\ddot{\theta} + V(\theta,\dot{\theta}) + G(\theta) = \tau$$

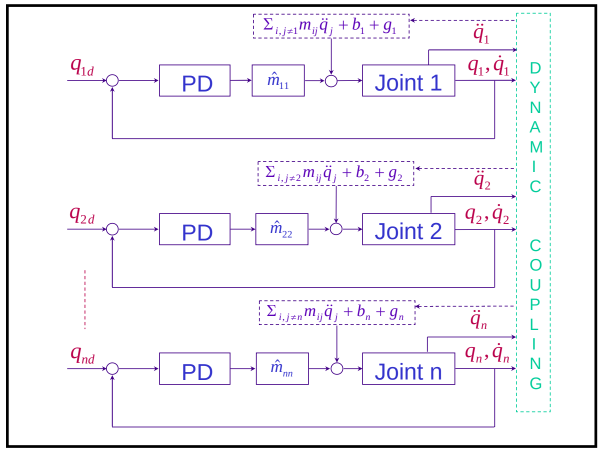
$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} + \begin{pmatrix} m_{112} \\ 0 \end{pmatrix}\begin{pmatrix} \dot{\theta}_1 & \dot{\theta}_2 \end{pmatrix} + \begin{pmatrix} 0 & m_{122} \\ -\frac{m_{112}}{2} & 0 \end{pmatrix}\begin{pmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \end{pmatrix} + \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}$$

$$m_{11}\ddot{\theta}_1 + m_{12}\ddot{\theta}_2 + m_{112}\dot{\theta}_1\dot{\theta}_2 + m_{122}\dot{\theta}_2^2 + G_1 = \tau_1$$

$$m_{22}\ddot{\theta}_2 + m_{21}\ddot{\theta}_1 - \frac{m_{112}}{2}\dot{\theta}_1^2 + G_2 = \tau_2$$







PD Control Stability

$$M(q)\ddot q + B(q)[\dot q\dot q] + C(q)[\dot q^2] + G(\theta) = \tau$$

$$\tau = -k_p(q - q_d) - k_v\dot q$$

$$V_d = 1/2k_p(q - q_d)^2$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot q}\right) - \frac{\partial K}{\partial q} + \frac{\partial V_s}{\partial q} = \not{\tau}\,\frac{\partial V_d}{\partial q} - k_v\dot q$$

PD Control Stability

$$A(q)\ddot q + B(q)[\dot q\dot q] + C(q)[\dot q^2] + g(\theta) = \Gamma$$

$$\Gamma = -k_p(q - q_d) - k_v\dot q$$

$$V_d = 1/2k_p(q - q_d)^2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q}\right) - \frac{\partial T}{\partial q} + \frac{\partial (V_s - V_d)}{\partial q} = \Gamma_s$$

$\Gamma_s = -k_v\dot q$ with $\Gamma_s^T\dot q < 0$ for $\dot q \neq 0;\ k_v > 0$

Performance

High Gains → better disturbance rejection

Gains are limited by
- structural flexibilities
- time delays (actuator-sensing)
- sampling rate

$\omega_n \leq \frac{\omega_{res}}{2}$ ← lowest structural flexibility

$\omega_n \leq \frac{\omega_{delay}}{3}$ ← largest delay $\left(\frac{2\pi}{\tau_{delay}}\right)$

$\omega_n \leq \frac{\omega_{sampling-rate}}{5}$

Nonlinear Dynamic Decoupling

$$A(q)\ddot q + b(q,\dot q) + g(q) = \Gamma$$

where $\Gamma = \hat A(q)\Gamma^* + \hat b(q,\dot q) + \hat g(q)$

$$\Gamma^* = I_n\ddot q_d - k_v(\dot q - \dot q_d) - k_p(q - q_d)$$