This sheet summarizes some useful formulas for dynamics. For simplicity of notation, all velocities, accelerations, angular velocities, and angular accelerations are assumed to be with respect to (measured in) the world *Newtonian* frame, N. Note "with respect to" is **not** the same as "expressed in". The first has to do with actual physical motion and the reference frame of the observer; the second has to do with the choice of variables and unit vectors.

Our notation assumes an arbitrary frame F has an origin F_o and orthonormal unit vectors $\widehat{\mathbf{f}}_{\mathrm{x}}$, $\widehat{\mathbf{f}}_{\mathrm{y}}$, $\widehat{\mathbf{f}}_{\mathrm{z}}$ fixed in it.

1 Force Models

1.1 Translational Spring-Damper

For points Q and P connected by a translational spring-damper, the force **on** Q **from** P is given by:

Spring: $\vec{\mathbf{F}}^{Q/P}_{\text{spring}} = -k(L - L_n) * \widehat{\mathbf{u}}^{Q/P}$, where $L = \left|\vec{\mathbf{r}}^{\,Q/P}\right|$ and L_n is the spring's natural length.

Damper: $\vec{\mathbf{F}}^{Q/P}_{\text{damper}} = -b\dot{L} * \widehat{\mathbf{u}}^{Q/P}$, where $\dot{L} = \vec{\mathbf{v}}^{Q/P} \cdot \widehat{\mathbf{u}}^{Q/P}$.

Note: The the unit-vector pointing to Q from P is $\widehat{\mathbf{u}}^{Q/P} = \frac{1}{L}(\vec{\mathbf{r}}^{\,Q/P})$.
Hint: Instead of L, use $L + \epsilon$ to avoid divide-by-zero issues.

1.2 Air

Translational air resistance on body B is often modeled as $\vec{\mathbf{F}}^{B}_{\text{air}} = -b_{\text{air},1} * {}^{\text{air}}\vec{\mathbf{v}}^{B_{\text{cm}}}$.
Rotational air resistance on body B is often modeled as $\vec{\mathbf{T}}^{B}_{\text{air}} = -b_{\text{air},2} * {}^{\text{air}}\vec{\boldsymbol{\omega}}^{B}$.

1.3 Gravity

The effect of gravity is often modeled as a single force acting at the center-of-mass of a body (or particle): $\vec{\mathbf{F}}^{B}_{\text{g}} = mg *$ (unit-vector pointing "down").

2 Kinematics

2.1 Translation

Represent the vector from N_o to particle/body center B_{cm} using measures along basis vectors $\widehat{\mathbf{n}}_{\mathrm{x}}$, $\widehat{\mathbf{n}}_{\mathrm{y}}$, $\widehat{\mathbf{n}}_{\mathrm{z}}$:

$\vec{\mathbf{r}}^{\,B_{\text{cm}}/N_o} = x\ \widehat{\mathbf{n}}_{\mathrm{x}} + y\ \widehat{\mathbf{n}}_{\mathrm{y}} + z\ \widehat{\mathbf{n}}_{\mathrm{z}}$ Position vector to B_{cm} from N_o.
$\vec{\mathbf{v}}^{B_{\text{cm}}} = \dot{x}\ \widehat{\mathbf{n}}_{\mathrm{x}} + \dot{y}\ \widehat{\mathbf{n}}_{\mathrm{y}} + \dot{z}\ \widehat{\mathbf{n}}_{\mathrm{z}}$ Velocity of B_{cm} (expressed in $\widehat{\mathbf{n}}_{\mathrm{xyz}}$).
$\vec{\mathbf{a}}^{B_{\text{cm}}} = \ddot{x}\ \widehat{\mathbf{n}}_{\mathrm{x}} + \ddot{y}\ \widehat{\mathbf{n}}_{\mathrm{y}} + \ddot{z}\ \widehat{\mathbf{n}}_{\mathrm{z}}$ Acceleration of B_{cm} (expressed in $\widehat{\mathbf{n}}_{\mathrm{xyz}}$).

For the rigid body case, the position and velocity of the attachment point Q on the edge of the sphere is:

$$\vec{\mathbf{r}}^{\,Q/N_o} = \vec{\mathbf{r}}^{\,B_{\text{cm}}/N_o} + \vec{\mathbf{r}}^{\,Q/B_{\text{cm}}} = x\ \widehat{\mathbf{n}}_{\mathrm{x}} + y\ \widehat{\mathbf{n}}_{\mathrm{y}} + z\ \widehat{\mathbf{n}}_{\mathrm{z}} + r\ \widehat{\mathbf{b}}_{\mathrm{z}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\widehat{\mathbf{n}}_{\mathrm{xyz}}} + {}^{\mathrm{n}}R^{\mathrm{b}} * \begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix}_{\widehat{\mathbf{b}}_{\mathrm{xyz}}}$$

$$\vec{\mathbf{v}}^{Q} = \vec{\mathbf{v}}^{B_{\text{cm}}} + \vec{\boldsymbol{\omega}}^{B} \times \vec{\mathbf{r}}^{\,Q/B_{\text{cm}}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}_{\widehat{\mathbf{n}}_{\mathrm{xyz}}} + \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}_{\widehat{\mathbf{n}}_{\mathrm{xyz}}} \times \left({}^{\mathrm{n}}R^{\mathrm{b}} * \begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix}_{\widehat{\mathbf{b}}_{\mathrm{xyz}}} \right)$$

2.2 Rotation

Particles have no orientation, so there is no need to do any rotational forces or kinematics.

For a rigid body, represent the angular velocity of body B as: $\vec{\boldsymbol{\omega}}^{B} = \omega_x\ \widehat{\mathbf{n}}_{\mathrm{x}} + \omega_y\ \widehat{\mathbf{n}}_{\mathrm{y}} + \omega_z\ \widehat{\mathbf{n}}_{\mathrm{z}} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}_{\widehat{\mathbf{n}}_{\mathrm{xyz}}}$

3 Equations of Motion

3.1 Translation

Newton's 2nd Law relates forces on a body B (or particle Q) to the acceleration of the body's center of mass:

$$\vec{\mathbf{F}}^{B} = m^{B} * \vec{\mathbf{a}}^{B_{\text{cm}}} \qquad\qquad \vec{\mathbf{F}}^{Q} = m^{Q} * \vec{\mathbf{a}}^{Q}$$

3.2 Rotation

Euler's 3D rigid body equation relates the resultant moment on body B to its angular acceleration.
(Without diving into the subtleties, in this case we can treat moments and torques as the same thing.)

$$\vec{\mathbf{M}}^{B/B_{\text{cm}}} = \vec{\vec{\mathbf{I}}}^{B/B_{\text{cm}}} \cdot \vec{\boldsymbol{\alpha}}^{B} + \vec{\boldsymbol{\omega}}^{B} \times (\vec{\vec{\mathbf{I}}}^{B/B_{\text{cm}}} \cdot \vec{\boldsymbol{\omega}}^{B})$$

In the <u>*very special case*</u> when the inertia matrix of a body can be written in terms of a scalar I times the identity matrix (e.g. for a sphere or box), the full form of Euler's 3D rigid body equation collapses to:

$$\vec{\mathbf{M}}^{B/B_{\text{cm}}} = I * \vec{\boldsymbol{\alpha}}^{B} \qquad\qquad \vec{\mathbf{M}}^{B/B_{\text{cm}}} = \frac{2}{5} m r^2 * \vec{\boldsymbol{\alpha}}^{B} \quad \text{(for a sphere)}$$

Note the moment about B_{cm} due to a force applied at point Q is: $\vec{\mathbf{M}}^{\vec{\mathbf{F}}^{Q}/B_{\text{cm}}} = \vec{\mathbf{r}}^{\,Q/B_{\text{cm}}} \times \vec{\mathbf{F}}^{Q}$.

4 Numerical Integration

A simple (but poor-performing) integration scheme is explicit forward-Euler.

4.1 Translation

$$\begin{aligned} \vec{\mathbf{v}}^{B_{\text{cm}}}[k+1] &= \vec{\mathbf{v}}^{B_{\text{cm}}}[k] + \vec{\mathbf{a}}^{B_{\text{cm}}}[k] * \Delta t \\ \vec{\mathbf{r}}^{\,B_{\text{cm}}/N_o}[k+1] &= \vec{\mathbf{r}}^{\,B_{\text{cm}}/N_o}[k] + \vec{\mathbf{v}}^{B_{\text{cm}}}[k] * \Delta t \end{aligned}$$

4.2 Rotation

$$\vec{\boldsymbol{\omega}}^{B}[k+1] = \vec{\boldsymbol{\omega}}^{B}[k] + \vec{\boldsymbol{\alpha}}^{B}[k] * \Delta t$$

There are multiple ways to go from angular velocity to an orientation representation:

- If you use a quaternion $q = [w\ x\ y\ z]$, you can do:

  $$q[k+1] = q[k] + \Delta t * (\frac{1}{2} * H * \vec{\boldsymbol{\omega}}^{B}[k])$$

  where

  $$H = \begin{bmatrix} -x & -y & -z \\ w & -z & y \\ z & w & -x \\ -y & x & w \end{bmatrix}$$

  After each step, you should normalize the quaternion by dividing by its magnitude $\sqrt{w^2 + x^2 + y^2 + z^2}$.

- You can use your $[\omega_x\ \ \omega_y\ \ \omega_z] * \Delta t$ as an axis-angle representation, convert to a rotation matrix, and multiply by your previous rotation:

  $${}^{\text{n}}R^{\text{b}}[k+1] = R_{\text{axis-angle}}[k] * {}^{\text{n}}R^{\text{b}}[k]$$