TABLE II
EXPERIMENTAL RESULTS OF CALIBRATING THE CAMERA POSITION PARAMETERS

	Measured Position Parameters	Computed Position Parameters	Average Error Rate (%)
1	(94.5, −56.0, 16.5)	(89.83, −53.49, 17.02)	4.2
2	(198.0, −160.0, 15.7)	(198.23, −158.74, 17.16)	3.4
3	(297.5, −258.0, 15.5)	(295.29, −259.35, 15.72)	0.9
4	(92.0, −51.0, 30.0)	(91.11, −52.37, 30.54)	1.8
5	(196.5, −150.0, 30.0)	(195.26, −152.42, 30.63)	1.4
6	(298.5, −250.0, 30.0)	(298.90, −250.99, 30.96)	1.2
7	(99.0, −43.0, 41.0)	(100.72, −42.34, 42.24)	2.0
8	(204.0, −143.0, 41.5)	(208.72, −144.72, 43.39)	2.7
9	(303.0, −245.0, 41.5)	(307.60, −246.73, 42.37)	1.4

TABLE III
SIMULATION RESULTS

Noise Deviation (pixel)	Pan Error (degree)	Tilt Error (degree)	Swing Error (degree)	Focal Length Error (pixel)	Distance Error (cm)	Distance Error Rate (%)
0.25	0.39	0.07	0.27	2.48	0.98	0.92
0.50	0.77	0.14	0.55	4.99	1.97	1.85
0.75	1.16	0.21	0.82	7.52	2.95	2.78
1.00	1.54	0.28	1.09	10.07	3.94	3.71
1.50	2.09	0.42	1.54	15.19	5.53	5.20
2.00	2.52	0.55	1.95	19.97	6.89	6.48
2.50	2.97	0.72	2.35	26.25	8.41	7.91
3.00	3.37	0.89	2.71	32.86	9.89	9.30
3.50	3.71	1.01	3.03	38.29	11.09	10.43
4.00	3.54	1.14	3.21	44.30	11.58	10.89

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Least-Squares Estimation of Transformation Parameters Between Two Point Patterns

Shinji Umeyama

Abstract— **In many applications of computer vision, the following problem is encountered. Two point patterns (sets of points) $\{x_i\}$ and $\{y_i\}$; $i = 1, 2, \cdots, n$ are given in *m*-dimensional space, and we want to find the similarity transformation parameters (rotation, translation, and scaling) that give the least mean squared error between these point patterns. Recently Arun *et al.* and Horn *et al.* have presented a solution of this problem. Their solution, however, sometimes fails to give a correct rotation matrix and gives a reflection instead when the data is severely corrupted. The theorem given in this correspondence is a strict solution of the problem, and it always gives the correct transformation parameters even when the data is corrupted.**

Index Terms— **Absolute orientation problem, computer vision, least-squares, motion estimation, singular value decomposition.**

I. INTRODUCTION

In computer vision applications, we sometimes encounter the following mathematical problem. We are given two point patterns (sets of points) $\{x_i\}$ and $\{y_i\}$; $i = 1, 2, \cdots, n$ in m-dimensional space, and we want to find the similarity transformation parameters (R: rotation, t: translation, and c: scaling) giving the minimum value

Manuscript received October 29, 1989; revised August 30, 1990. Recommended for acceptance by C. Brown.

The author is with the Mathematical Informatics Section, Information Science Division, Electrotechnical Laboratory, 1-1-4 Umezono, Tsukuba-shi, Ibaraki 305, Japan.

IEEE Log Number 9041531.

0162–8828/91/0400–0376$01.00 © 1991 IEEE

of the mean squared error $e^2(R, \boldsymbol{t}, c)$ of these two point patterns.

$$e^2(R, \boldsymbol{t}, c) = \frac{1}{n}\sum_{i=1}^{n} \|\boldsymbol{y}_i - (cR\boldsymbol{x}_i + \boldsymbol{t})\|^2 \tag{1}$$

The dimensionality m is usually 2 or 3.

This problem is sometimes called the absolute orientation problem [1], and an iterative algorithms for finding the solution [2] and a noniterative algorithm based on quaternions [3] are proposed for a 3-D problem. A good reference can be found in [1]. Recently, Arun *et al.* [4] and Horn *et al.* [1] have presented a solution of this problem, which is based on the singular value decomposition of a covariance matrix of the data. Their solution, however, sometimes fails to give a correct rotation matrix and gives a reflection instead ($\det(R) = -1$) when the data is severely corrupted.

The theorem given in this correspondence is a strict solution of the problem, and it is derived by refining Arun's result. The theorem always gives the correct transformation parameters even when the data is corrupted.

II. Least-Squares Estimation of Transformation Parameters

In this section, we show a theorem which gives the least-squares estimation of similarity transformation parameters between two point patterns. Before showing the theorem, we prove a lemma, which gives the least-squares estimation of rotation parameters. This lemma is the main result of this correspondence.

Lemma: Let A and B be $m \times n$ matrices, and R an $m \times m$ rotation matrix, and UDV^T a singular value decomposition of AB^T ($UU^T = VV^T = I$, $D = \mathrm{diag}(d_i), d_1 \geq d_2 \geq \cdots \geq d_m \geq 0$). Then the minimum value of $\|A - RB\|^2$ with respect to R is

$$\min_R \|A - RB\|^2 = \|A\|^2 + \|B\|^2 - 2\mathrm{tr}(DS) \tag{2}$$

where

$$S = \begin{cases} I & \text{if } \det(AB^T) \geq 0 \\ \mathrm{diag}(1, 1, \cdots, 1, -1) & \text{if } \det(AB^T) < 0. \end{cases} \tag{3}$$

When $\mathrm{rank}(AB^T) \geq m - 1$, the optimum rotation matrix R which achieves the above minimum value is uniquely determined.

$$R = USV^T \tag{4}$$

where S in (4) must be chosen as

$$S = \begin{cases} I & \text{if } \det(U)\det(V) = 1 \\ \mathrm{diag}(1, 1, \cdots, 1, -1) & \text{if } \det(U)\det(V) = -1 \end{cases} \tag{5}$$

when $\det(AB^T) = 0$ ($\mathrm{rank}(AB^T) = m - 1$).

Proof of Lemma: Define an objective function F as

$$F = \|A - RB\|^2 + \mathrm{tr}\left(L\left(R^TR - I\right)\right) + g\{\det(R) - 1\} \tag{6}$$

where g is a Lagrange multiplier and L is a symmetric matrix of Lagrange multipliers. The second and third term of F represent the conditions for R to be an orthogonal and proper rotation matrix respectively. Partial differentiations of F with respect to R, $\boldsymbol{t}$, and c lead to the following system of equations [5].

$$\frac{\partial F}{\partial R} = -2AB^T + 2RBB^T + 2RL + gR = 0 \tag{7}$$

$$\frac{\partial F}{\partial L} = R^TR - I = 0 \tag{8}$$

$$\frac{\partial F}{\partial g} = \det(R) - 1 = 0 \tag{9}$$

where we used

$$\frac{\partial}{\partial R}\det(R) = \mathrm{adj}\left(R^T\right) = \det\left(R^T\right)\left(R^T\right)^{-1} = R \tag{10}$$

since R is a rotation matrix ($\mathrm{adj}(R^T)$ is an adjoint matrix of R^T). From (7),

$$RL' = AB^T, \quad \text{where } L' = BB^T + L + \frac{1}{2}gI. \tag{11}$$

By transposing the both sides of (11), we obtain the following equation (note that L' is symmetric).

$$L'R^T = BA^T \tag{12}$$

If we multiply each side of (11) with each side of (12), respectively, (13) is obtained since $R^TR = I$.

$$L'^2 = BA^TAB^T = VD^2V^T \tag{13}$$

Obviously L' and L'^2 are commutative ($L'L'^2 = L'^2L'$), hence both can be reduced to diagonal forms by the same orthogonal matrix [6]. Thus we can write

$$L' = VDSV^T, \tag{14}$$

where $S = \mathrm{diag}(s_i)$, $s_i = 1$, or -1.

Now, from (14),

$$\begin{aligned}\det(L') &= \det\left(VDSV^T\right) \\ &= \det(V)\det(D)\det(S)\det\left(V^T\right) \\ &= \det(D)\det(S).\end{aligned} \tag{15}$$

On the other hand, from (11)

$$\begin{aligned}\det(L') &= \det\left(R^TAB^T\right) \\ &= \det\left(R^T\right)\det\left(AB^T\right) \\ &= \det\left(AB^T\right).\end{aligned} \tag{16}$$

Thus,

$$\det(D)\det(S) = \det\left(AB^T\right). \tag{17}$$

Since singular values are nonnegative, $\det(D) = d_1d_2\cdots d_m \geq 0$. Hence $\det(S)$ must be equal to 1 when $\det(AB^T) > 0$, and -1 when $\det(AB^T) < 0$.

Next, extremum values of $\|A - RB\|^2$ is derived as follows: from (11) we have

$$\begin{aligned}\|A - RB\|^2 &= \|A\|^2 + \|B\|^2 - 2\mathrm{tr}\left(AB^TR^T\right) \\ &= \|A\|^2 + \|B\|^2 - 2\mathrm{tr}\left(R^TAB^T\right) \\ &= \|A\|^2 + \|B\|^2 - 2\mathrm{tr}(L').\end{aligned} \tag{18}$$

Substituting (14) into (18), we have

$$\begin{aligned}\|A - RB\|^2 &= \|A\|^2 + \|B\|^2 - 2\mathrm{tr}\left(VDSV^T\right) \\ &= \|A\|^2 + \|B\|^2 - 2\mathrm{tr}(DS) \\ &= \|A\|^2 + \|B\|^2 - 2(d_1s_1 + d_2s_2 + \cdots + d_ms_m).\end{aligned} \tag{19}$$

Thus, the minimum value of $\|A - RB\|^2$ is obviously achieved when $s_1 = s_2 = \cdots s_m = 1$ if $\det(AB^T) \geq 0$, and $s_1 = s_2 = \cdots s_{m-1} = 1$, $s_m = -1$ if $\det(AB^T) < 0$. This concludes the first half of the lemma.

Next, we determine a rotation matrix R achieving the above minimum value. When $\mathrm{rank}(AB^T) = m$, L' is nonsingular, thus it has its inverse $L'^{-1} = \left(VDSV^T\right)^{-1} = VS^{-1}D^{-1}V^T = VD^{-1}SV^T$ (note that $S^{-1} = S$, $SD^{-1} = D^{-1}S$). Therefore, from (11) we have

$$R = AB^T L'^{-1} = UDV^T VD^{-1}SV^T = USV^T. \quad (20)$$

Finally, when $\operatorname{rank}(AB^T) = m - 1$, from (11), (14)

$$RVDSV^T = UDV^T. \quad (21)$$

Multiplying V by both sides of (21) from the right and using $DS = D$ (since $d_m = 0$ and $s_1 = s_2 = \cdots s_{m-1} = 1$),

$$RVD = UD \quad (22)$$

is obtained. If we define an orthogonal matrix Q as follows:

$$Q = U^T RV \quad (23)$$

we have

$$QD = D. \quad (24)$$

Let the column vectors of Q be $\boldsymbol{q}_1, \boldsymbol{q}_2, \cdots, \boldsymbol{q}_m$ ($Q = [\boldsymbol{q}_1, \boldsymbol{q}_2, \cdots, \boldsymbol{q}_m]$). The following equations are obtained by comparing both sides of (24).

$$d_i \boldsymbol{q}_i = d_i \boldsymbol{e}_i \qquad 1 \le i \le m - 1 \quad (25)$$

Hence,

$$\boldsymbol{q}_i = \boldsymbol{e}_i \qquad 1 \le i \le m - 1 \quad (26)$$

where $\boldsymbol{e}_i$ is a unit vector which has 1 as an ith element.

$$\boldsymbol{e}_i = (0, 0, \cdots, \overset{i}{1}, \cdots, 0)^T \quad (27)$$

The last column vector $\boldsymbol{q}_m$ of Q is orthogonal to all other vectors $\boldsymbol{q}_i$ ($1 \le i \le m - 1$) since Q is an orthogonal matrix. Thus we have

$$\boldsymbol{q}_m = \boldsymbol{e}_m \quad \text{or} \quad \boldsymbol{q}_m = -\boldsymbol{e}_m. \quad (28)$$

On the other hand,

$$\begin{aligned} \det(Q) &= \det\left(U^T\right) \det(R) \det(V) \\ &= \det(U) \det(V). \end{aligned} \quad (29)$$

Thus, $\det(Q) = 1$ if $\det(U)\det(V) = 1$ and $\det(Q) = -1$ if $\det(U)\det(V) = -1$. Therefore we have

$$\begin{aligned} R &= UQV^T \\ &= USV^T \end{aligned} \quad (30)$$

where

$$S = \begin{cases} I & \text{if } \det(U)\det(V) = 1 \\ \operatorname{diag}(1, 1, \cdots, 1, -1) & \text{if } \det(U)\det(V) = -1. \end{cases} \quad (31)$$

Q.E.D.

We can derive the following theorem using this lemma.

Theorem: Let $X = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n\}$ and $Y = \{\boldsymbol{y}_1, \boldsymbol{y}_2, \cdots, \boldsymbol{y}_n\}$ be corresponding point patterns in m-dimensional space. The minimum value ε^2 of the mean squared error

$$e^2(R, \boldsymbol{t}, c) = \frac{1}{n} \sum_{i=1}^{n} \|\boldsymbol{y}_i - (cR\boldsymbol{x}_i + \boldsymbol{t})\|^2 \quad (32)$$

of these two point patterns with respect to the similarity transformation parameters (R: rotation, $\boldsymbol{t}$: translation, and c: scaling) is given as follows:

$$\varepsilon^2 = \sigma_y^2 - \frac{\operatorname{tr}(DS)^2}{\sigma_x^2} \quad (33)$$

where

$$\boldsymbol{\mu}_x = \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{x}_i \quad (34)$$

$$\boldsymbol{\mu}_y = \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{y}_i \quad (35)$$

$$\sigma_x^2 = \frac{1}{n} \sum_{i=1}^{n} \|\boldsymbol{x}_i - \boldsymbol{\mu}_x\|^2 \quad (36)$$

$$\sigma_y^2 = \frac{1}{n} \sum_{i=1}^{n} \left\|\boldsymbol{y}_i - \boldsymbol{\mu}_y\right\|^2 \quad (37)$$

$$\Sigma_{xy} = \frac{1}{n} \sum_{i=1}^{n} (\boldsymbol{y}_i - \boldsymbol{\mu}_y)(\boldsymbol{x}_i - \boldsymbol{\mu}_x)^T \quad (38)$$

and let a singular value decomposition of Σ_{xy} be UDV^T ($D = \operatorname{diag}(d_i)$, $d_1 \ge d_2 \ge \cdots \ge d_m \ge 0$), and

$$S = \begin{cases} I & \text{if } \det(\Sigma_{xy}) \ge 0 \\ \operatorname{diag}(1, 1, \cdots, 1, -1) & \text{if } \det(\Sigma_{xy}) < 0. \end{cases} \quad (39)$$

Σ_{xy} is a covariance matrix of X and Y, $\boldsymbol{\mu}_x$ and $\boldsymbol{\mu}_y$ are mean vectors of X and Y, and σ_x^2 and σ_y^2 are variances around the mean vectors of X and Y, respectively.

When rank $(\Sigma_{xy}) \ge m - 1$, the optimum transformation parameters are determined uniquely as follows:

$$R = USV^T \quad (40)$$

$$\boldsymbol{t} = \boldsymbol{\mu}_y - cR\boldsymbol{\mu}_x \quad (41)$$

$$c = \frac{1}{\sigma_x^2} \operatorname{tr}(DS) \quad (42)$$

where S in (40) must be chosen as

$$S = \begin{cases} I & \text{if } \det(U)\det(V) = 1 \\ \operatorname{diag}(1, 1, \cdots, 1, -1) & \text{if } \det(U)\det(V) = -1 \end{cases} \quad (43)$$

when $(\operatorname{rank}(\Sigma_{xy}) = m - 1)$.

Proof: We represent the point sets X, Y, by $m \times n$ matrices $X = [\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n]$, $Y = [\boldsymbol{y}_1, \boldsymbol{y}_2, \cdots, \boldsymbol{y}_n]$, respectively. Then, $e^2(R, \boldsymbol{t}, c)$ in (32) is reformulated as follows:

$$e^2(R, \boldsymbol{t}, c) = \frac{1}{n} \left\|Y - cRX - \boldsymbol{t}\boldsymbol{h}^T\right\|^2 \quad (44)$$

where

$$\boldsymbol{h} = (1, 1, \cdots, 1)^T. \quad (45)$$

Here, we introduce an $n \times n$ normalization matrix $K = I - (1/n)\boldsymbol{h}\boldsymbol{h}^T$ ($K^2 = K^T = K$). Using this matrix, the characteristics in (36)–(38) are written as follows:

$$\sigma_x^2 = \frac{1}{n} \|XK\|^2 \quad (46)$$

$$\sigma_y^2 = \frac{1}{n} \|YK\|^2 \quad (47)$$

$$\Sigma_{xy} = \frac{1}{n} YKX^T. \quad (48)$$

Moreover, if we use the following equations,

$$X = XK + \frac{1}{n}X\boldsymbol{h}\boldsymbol{h}^T \tag{49}$$

$$Y = YK + \frac{1}{n}Y\boldsymbol{h}\boldsymbol{h}^T \tag{50}$$

$e^2(R, \boldsymbol{t}, c)$ is further reformulated as follows:

$$\begin{aligned} e^2(R,\boldsymbol{t},c) &= \frac{1}{n}\left\| YK + \frac{1}{n}Y\boldsymbol{h}\boldsymbol{h}^T - cRXK - \frac{c}{n}RX\boldsymbol{h}\boldsymbol{h}^T - \boldsymbol{t}\boldsymbol{h}^T \right\|^2 \\ &= \frac{1}{n}\left\| YK - cRXK + \left(\frac{1}{n}Y\boldsymbol{h} - \frac{c}{n}RX\boldsymbol{h} - \boldsymbol{t}\right)\boldsymbol{h}^T \right\|^2 \\ &= \frac{1}{n}\left\| YK - cRXK - \boldsymbol{t}'\boldsymbol{h}^T \right\|^2 \\ &= \frac{1}{n}\Big\{ \|YK - cRXK\|^2 + \left\|\boldsymbol{t}'\boldsymbol{h}^T\right\|^2 \\ &\quad - 2\mathrm{tr}\Big(K\left(Y^T - cX^TR^T\right)\boldsymbol{t}'\boldsymbol{h}^T\Big)\Big\} \end{aligned} \tag{51}$$

where

$$\boldsymbol{t}' = -\frac{1}{n}Y\boldsymbol{h} + \frac{c}{n}RX\boldsymbol{h} + \boldsymbol{t}. \tag{52}$$

Since we can show the following equations

$$\begin{aligned} \mathrm{tr}\Big(K\left(Y^T - cX^TR^T\right)\boldsymbol{t}'\boldsymbol{h}^T\Big) &= \mathrm{tr}\Big(\boldsymbol{h}^T\left(I - \frac{1}{n}\boldsymbol{h}\boldsymbol{h}^T\right) \\ &\qquad \times \left(Y^T - cX^TR^T\right)\boldsymbol{t}'\Big) \\ &= \mathrm{tr}\Big(\left(\boldsymbol{h}^T - \boldsymbol{h}^T\right) \\ &\qquad \times \left(Y^T - cX^TR^T\right)\boldsymbol{t}'\Big) \\ &= 0 \end{aligned} \tag{53}$$

$$\left\|\boldsymbol{t}'\boldsymbol{h}^T\right\|^2 = n\|\boldsymbol{t}'\|^2 \tag{54}$$

we have

$$e^2(R,\boldsymbol{t},c) = \frac{1}{n}\|YK - cRXK\|^2 + \|\boldsymbol{t}'\|^2. \tag{55}$$

From this equation, $\boldsymbol{t}'$ must be equal to 0 in order to minimize $e^2(R, \boldsymbol{t}, c)$, that is,

$$\boldsymbol{t} = \frac{1}{n}Y\boldsymbol{h} - \frac{c}{n}RX\boldsymbol{h} = \boldsymbol{\mu}_y - cR\boldsymbol{\mu}_x. \tag{56}$$

Next, when UDV^T is a singular value decomposition of $\Sigma_{xy} = (1/n)YKX^T$, a singular value decomposition of $YK(cXK)^T = cYKK^TX^T = cYKX^T$ is $cnUDV^T$. Thus, the minimum value $\varepsilon^2(c)$ of $(1/n)\|YK - cRXK\|^2$ with respect to R is given from the lemma as follows:

$$\begin{aligned} \varepsilon^2(c) &= \frac{1}{n}\{\|YK\|^2 + \|cXK\|^2 - 2\mathrm{tr}(cnDS)\} \\ &= \sigma_y^2 + c^2\sigma_x^2 - 2c\mathrm{tr}(DS) \end{aligned} \tag{57}$$

where

$$S = \begin{cases} I & \text{if } \det(\Sigma_{xy}) \geq 0 \\ \mathrm{diag}(1, 1, \cdots, 1, -1) & \text{if } \det(\Sigma_{xy}) < 0. \end{cases} \tag{58}$$

Also from the above lemma, if $\mathrm{rank}(\Sigma_{xy}) \geq m - 1$,

$$R = USV^T \tag{59}$$

where S must be chosen as

$$S = \begin{cases} I & \text{if } \det(U)\det(V) = 1 \\ \mathrm{diag}(1, 1, \cdots, 1, -1) & \text{if } \det(U)\det(V) = -1 \end{cases} \tag{60}$$

when $\mathrm{rank}(\Sigma_{xy}) = m - 1$.

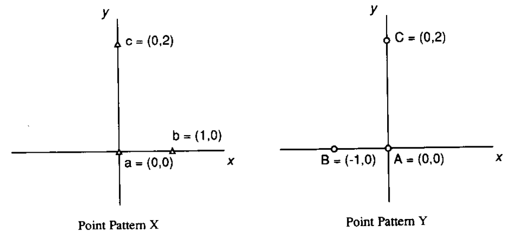

Fig. 1. Two point patterns X and Y consisting of three points in two-dimensional space.

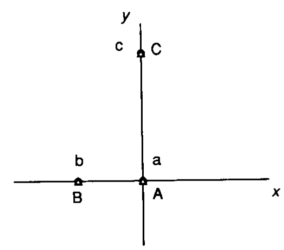

Fig. 2. The matching result without the proper condition of R.

Finally, since $\varepsilon^2(c)$ is a quadratic form of c, the minimum value of $\varepsilon^2(c)$ is obviously achieved when

$$c = \frac{\mathrm{tr}(DS)}{\sigma_x^2} \tag{61}$$

and the minimum value ε^2 is

$$\begin{aligned} \varepsilon^2 &= \sigma_y^2 + \left\{\frac{\mathrm{tr}(DS)}{\sigma_x^2}\right\}^2 \sigma_x^2 - 2\left\{\frac{\mathrm{tr}(DS)}{\sigma_x^2}\right\}\mathrm{tr}(DS) \\ &= \sigma_y^2 - \frac{\mathrm{tr}(DS)^2}{\sigma_x^2}. \end{aligned} \tag{62}$$

This concludes the theorem. Q.E.D.

III. Numerical Example

Now we show a very simple numerical example of the absolute orientation problem, where Arun and Horn's method gives a reflection, while the proposed method successfully gives a rotation.

Fig. 1 shows two point patterns X and Y consisting of three points ((a, b, c) in X, and (A, B, C) in Y) in two-dimensional space, respectively. Here we assume that a point a in X is matched with a point A in Y, b to B, and c to C. Then Arun and Horn's method gives the following transformation parameters.

$$R = \begin{pmatrix} -1.0 & 0.0 \\ 0.0 & 1.0 \end{pmatrix}, \quad \boldsymbol{t} = \begin{pmatrix} 0.0 \\ 0.0 \end{pmatrix}, \quad c = 1.0 \tag{63}$$

The least mean squared error $\varepsilon^2 = 0.0$, and the point pattern Y and the transformed point pattern of X is shown in Fig. 2. The obtained transformation gives a perfect matching $(\varepsilon^2 = 0.0)$. However, it obviously represents a reflection. Thus, if a reflection is not allowed as a transformation between X and Y, their method fails to give an appropriate solution in this case.

On the other hand, the transformation parameters given by the proposed method is as follows.

$$R = \begin{pmatrix} 0.832 & 0.555 \\ -0.555 & 0.832 \end{pmatrix}, \quad \boldsymbol{t} = \begin{pmatrix} -0.800 \\ 0.400 \end{pmatrix}, \quad c = 0.721 \tag{64}$$

The least mean squared error $\varepsilon^2 = 0.533$, and the transformation

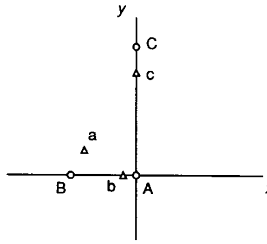

Fig. 3. The matching result with the proper condition of R.

result is shown in Fig. 3. This is the optimum transformation under the condition that R should be a real rotation matrix.

IV. Conclusion

We presented here a closed-form solution of the least-squares problem of the similarity transformation parameter estimation, using the singular value decomposition of a covariance matrix of the data. The solution is applicable to any dimensional problem, though the quaternion method is valid only for point patterns in three-dimensional space. The presented solution is considered to be a refinement of Arun and Horn's method, however it always gives the correct rotation matrix even when their method fails. Arun and Horn's result can be obtained if we set $S = I$ in (40) without regard to the sign of $\det(\Sigma_{xy})$. When more than two distinct points in two-dimensional space and more than three noncolinear points in three-dimensional space are given, the solution can determine the transformation parameters uniquely, since $\mathrm{rank}(\Sigma_{xy}) = m - 1$ in these cases.

In concluding the correspondence, we would like to mention that after we had submitted our original manuscript, it was brought to our attention by one of the reviewers that results similar to some of ours were mentioned independently by Holder *et al.* [7]. They point out that a rotation matrix of the solution of an absolute orientation problem must satisfy an equation similar to (11). The details, however, are not given in their paper.

Acknowledgment

The author wishes to thank the anonymous reviewers for their many valuable suggestions.

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The Topology of Locales and Its Effects on Position Uncertainty

David I. Havelock

***Abstract*—The precision to which the position of a target in a digital image can be estimated, may be analyzed by considering the possible digital representations of the target. Such an analysis leads to regions of indistinguishable target position, referred to as locales. By considering the density, distribution, and shape of these locales the available precision can be estimated. Previously, such analyses have presumed an absence of noise in the digital image. It is shown here how the noise tolerance for position estimation is affected by the topological properties of locales, such as locale connectivity, adjacency, and clustering.**

***Index Terms*—Image metrology, locales, noise, precision, registration, targets.**

I. Introduction

Subpixel position estimation for targets in digital images has been investigated for some time, particularly in regard to image registration. Recent articles have investigated geometric precision by means of enumerating the distinct digital representations of a target in digital images. By this approach, an expression has been derived for the number of distinct digital representations of a binary straight edge, when the edge has a known orientation [1]. Asymptotic expressions have been derived when neither orientation nor offset are known [2], [3]. Binary targets for optimal registration have been designed, based on maximizing the number of distinct digital representations [4], [5]. For more general target shapes, and abstract position parameters, graphical evaluation of registration precision and analytical bounds on precision have been developed based on parameter equivalence classes (*locales*) defined by the digital representations of the target [6].

Related work exists for the digital representations of line segments [7]–[11], arc [12] and circles [13], as well as analysis of precision for digitizing schemes [14]–[16], and position estimation algorithms [17]–[22], to list a few. The optimal position estimate, in regard to errors due to quantization and sampling, has been defined in a natural way as the center of the region (*locale*) corresponding to each digital representation of the target [23], [24]. In the analyses presented here, target *positional* uncertainty is considered, rather than the more commonly investigated image *intensity* errors. (The latter being exemplified, for example, by the excellent analysis in [14] which considers image intensity errors, rather than positional errors, due to combined quantization and sampling.)

One shortcoming of the method of analyzing geometric precision by enumerating the digital representations of a target has been the difficulty in dealing with noise in an analytically consistent manner. Typically, a noise-free analysis is developed, followed by a series of simulations to investigate the effects of noise in an empirical manner. This is a convenient approach to *infer* the validity of the noise-free analysis in a realistic noisy image. It would be better, however, to incorporate the noise within the formal analytical framework in the first place.

Here, the relationship between image noise and positional uncertainty is investigated in the context of discrete digital representations of a target. Image noise, which causes the observed pixel values to differ from those of the ideal model, can be expressed as an *error volume* in image space. Registration error due to positional

Manuscript received December 1, 1989; revised October 1, 1990. Recommended for acceptance by C. R. Dyer.

The author is with the Institute for Microstructural Sciences, N.R.C., Ottawa, Ont. K1A OR6, Canada.

IEEE Log Number 9041532.

0162-8828/91/0400-0380$01.00