CME342 / AA220 / CS238
Parallel Methods in Numerical Analysis

Domain Decomposition Methods I

Outline:

- Sign up for final project slot starting Monday.

- Overlapping methods.
- Nonoverlapping methods.

Domain Decomposition Methods

- Key Idea: *Divide and Conquer*
- Suitable for parallel computing.
- Overlapping Methods
  - ▷ Multiplicative: Classical Alternating Schwarz
  - ▷ Additive
- Nonoverlapping Methods
  - ▷ Substructuring Iterative Methods
- Two Level Methods / Convergence Analysis
- Reference: *Domain Decomposition*, Barry Smith, Petter Bjorstad, William Gropp.

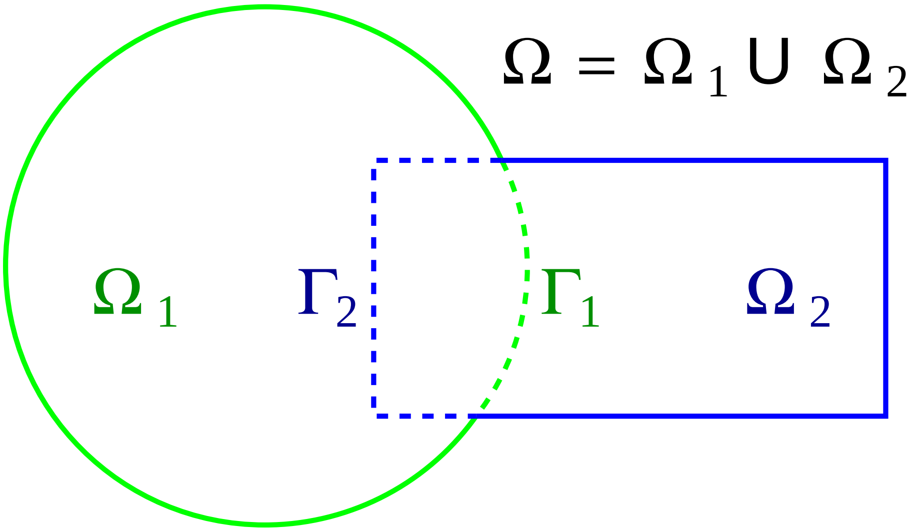
Scalability

- Ideal case 1: double # of procs, double speedup.
  - ▷ Amdahl's law.
  - ▷ Communication cost.

- Ideal case 2: double problem size, double execution time.
  - ▷ Not possible for solving linear systems using GE since $O(n^3)$.
  - ▷ Neither for Jacobi, GS whose complexity $=$ $O(n^2)$ nor CG $= O(n^{3/2})$.

- Scalable methods:
  domain decomposition, multigrid.

Model Problem: 2 Subdomains



Definitions:

$$
\begin{aligned}
\Omega &= \Omega_1 \cup \Omega_2 \\
\Gamma_i &= \partial\Omega_i \cap \Omega \\
&= \text{artificial interface/boundary} \\
u_i^k &= \text{approx. solution on } \bar{\Omega}_i \\
& \quad\ \text{after } k \text{ iterations} \\
u_1^k|_{\Gamma_2} &= \text{restriction of } u_1^k \text{ to } \Gamma_2 \\
u_2^k|_{\Gamma_1} &= \text{restriction of } u_2^k \text{ to } \Gamma_1
\end{aligned}
$$

Classical Alternating Schwarz (1869)

- Consider solving the following boundary value problem:

$$\begin{aligned} Lu &= f && \text{in } \Omega \\ u &= g && \text{on } \partial\Omega \end{aligned}$$

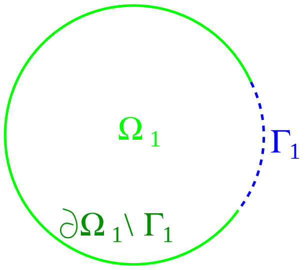
  e.g. $L = -\Delta$, the Laplacian operator.

- Alternating Schwarz algorithm:

  Starting with initial guess u_2^0 on Ω_2 (actually only need values of u_2^0 on Γ_1), for $k = 1, 2, \ldots$

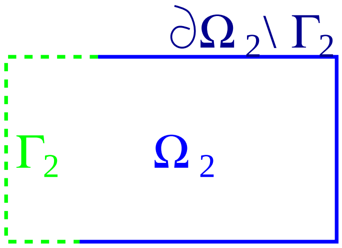
  (i) Solve for u_1^{k+1} in Ω_1:

$$\begin{cases} Lu_1^{k+1} = f & \text{in } \Omega_1 \\ u_1^{k+1} = g & \text{on } \partial\Omega_1 \backslash \Gamma_1 \\ u_1^{k+1} = u_2^k|_{\Gamma_1} & \text{on } \Gamma_1 \end{cases}$$



  (ii) Then, solve for u_2^{k+1} in Ω_2:

$$\begin{cases} Lu_2^{k+1} = f & \text{in } \Omega_2 \\ u_2^{k+1} = g & \text{on } \partial\Omega_2 \backslash \Gamma_2 \\ u_2^{k+1} = u_1^{k+1}|_{\Gamma_2} & \text{on } \Gamma_2 \end{cases}$$



Classical Alternating Schwarz (cont.)

- In each half step of alternating Schwarz:

  Solve the elliptic problem in Ω_i:

  $$Lu_i^{k+1} = f$$

  with given boundary values g on the true boundary $\partial\Omega_i \backslash \Gamma_i$:

  $$u_i^{k+1} = g \quad \partial\Omega_i \backslash \Gamma_i$$

  and latest approx. solution values on the artificial boundaries:

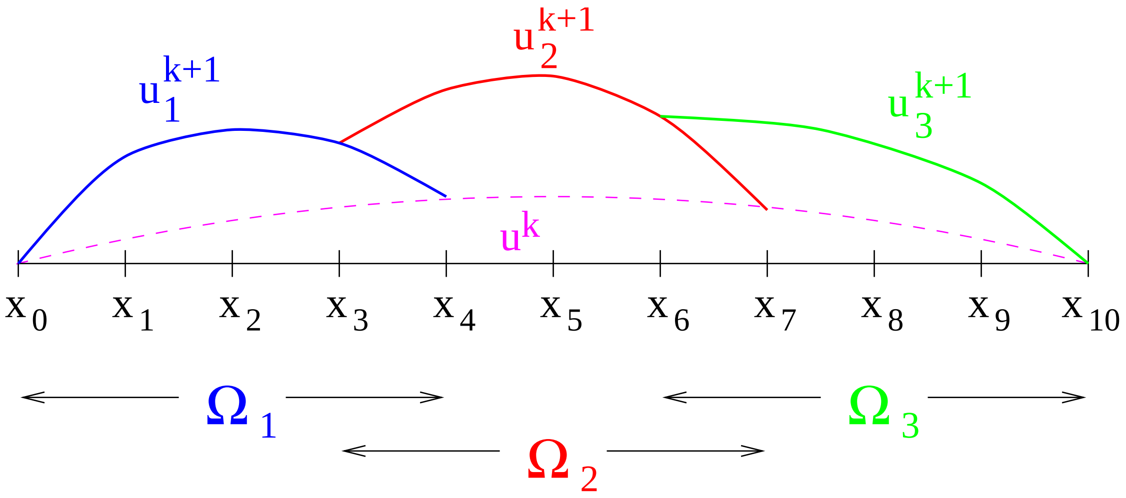
  $$u_1^{k+1} = u_2^k|_{\Gamma_1}, \quad u_2^{k+1} = u_1^{k+1}|_{\Gamma_2}$$

- Note that exact solution satisfies the two subdomain problems simultaneously
  $\Rightarrow$ Alternating Schwarz is consistent.

- May use other boundary conditions on the artificial boundaries Γ_i.
  e.g. Robin boundary condition on Γ_1:

  $$\alpha u_1^{k+1} + \beta \frac{\partial u_1^{k+1}}{\partial n} = \alpha u_2^k + \beta \frac{\partial u_2^k}{\partial n}$$

Example: 1D, 1 element overlap



- $\Omega_1 = (x_0, x_4)$, $\Omega_2 = (x_3, x_7)$, $\Omega_3 = (x_6, x_{10})$.
- (1) Solve for u_1^{k+1} in Ω_1:

$$\left\{ \begin{array}{rcll} \frac{d^2}{dx^2} u_1^{k+1} & = & f & \text{in } \Omega_1 \\ u_1^{k+1}(x_0) & = & 0 & \text{on } \partial\Omega_1 \backslash \Gamma_1 \\ u_1^{k+1}(x_4) & = & u_2^k(x_4) & \text{on } \Gamma_1 \end{array} \right.$$

Matrix form:

$$\begin{bmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{bmatrix} \begin{bmatrix} u_1^{k+1}(x_1) \\ u_1^{k+1}(x_2) \\ u_1^{k+1}(x_3) \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 + u_2^k(x_4) \end{bmatrix}$$

Example: 1D (cont.)

- (2) Solve for u_2^{k+1} in Ω_2:

$$\begin{cases} \frac{d^2}{dx^2} u_2^{k+1} = f & \text{in } \Omega_2 \\ u_2^{k+1}(x_3) = u_1^{k+1}(x_3) & \text{on } \Gamma_2 \\ u_2^{k+1}(x_7) = u_3^{k}(x_7) & \text{on } \Gamma_2 \end{cases}$$

Matrix form:

$$\begin{bmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{bmatrix} \begin{bmatrix} u_2^{k+1}(x_4) \\ u_2^{k+1}(x_5) \\ u_2^{k+1}(x_6) \end{bmatrix} = \begin{bmatrix} f_4 + u_1^{k+1}(x_3) \\ f_5 \\ f_6 + u_3^{k}(x_7) \end{bmatrix}$$

- (3) Solve for u_3^{k+1} in Ω_3:

$$\begin{bmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{bmatrix} \begin{bmatrix} u_3^{k+1}(x_7) \\ u_3^{k+1}(x_8) \\ u_3^{k+1}(x_9) \end{bmatrix} = \begin{bmatrix} f_7 + u_2^{k+1}(x_6) \\ f_8 \\ f_9 \end{bmatrix}$$

- $u^{k+1} = (u_1^{k+1}, u_2^{k+1}, u_3^{k+1})$.

Alternating Schwarz vs Block GS

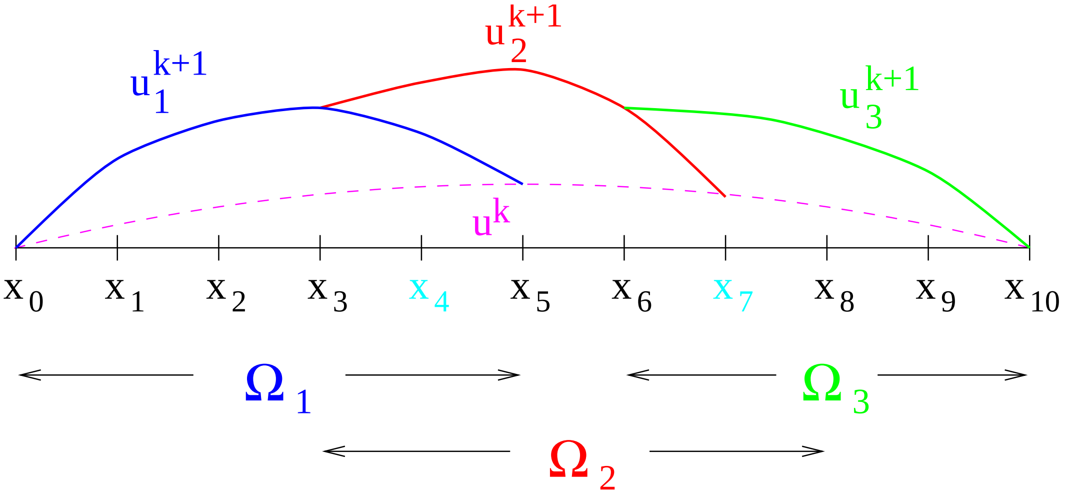
- Write A as 3×3 block:

$$A = \begin{bmatrix} 2 & -1 & & & & & & & \\ -1 & 2 & -1 & & & & & & \\ & -1 & 2 & -1 & & & & & \\ & & -1 & 2 & -1 & & & & \\ & & & -1 & 2 & -1 & & & \\ & & & & -1 & 2 & -1 & & \\ & & & & & -1 & 2 & -1 & \\ & & & & & & -1 & 2 & -1 \\ & & & & & & & -1 & 2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

- Block GS:

$$\begin{aligned} A_{11}x_{\vec{1}}^{k+1} &= f_{\vec{1}} - A_{12}x_{\vec{2}}^{k} - A_{13}x_{\vec{3}}^{k} \\ &= f_{\vec{1}} - A_{12}x_{\vec{2}}^{k} \\ &= \begin{bmatrix} f_1 \\ f_2 \\ f_3 + x_4^k \end{bmatrix} \\ A_{22}x_{\vec{2}}^{k+1} &= f_{\vec{2}} - A_{21}x_{\vec{1}}^{k+1} - A_{23}x_{\vec{3}}^{k} \\ &= \begin{bmatrix} f_4 + x_3^{k+1} \\ f_5 \\ f_6 + x_7^k \end{bmatrix} \\ A_{33}x_{\vec{3}}^{k+1} &= f_{\vec{3}} - A_{31}x_{\vec{1}}^{k+1} - A_{32}x_{\vec{2}}^{k+1} \\ &= \begin{bmatrix} f_7 + x_6^{k+1} \\ f_8 \\ f_9 \end{bmatrix} \end{aligned}$$

Example: 1D, 2 elements overlap



- (1) Solve for u_1^{k+1} in Ω_1:

$$\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix} \begin{bmatrix} u_1^{k+1}(x_1) \\ u_1^{k+1}(x_2) \\ u_1^{k+1}(x_3) \\ u_1^{k+1}(x_4) \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 + u_2^k(x_5) \end{bmatrix}$$

- (2) Solve for u_2^{k+1} in Ω_2:

$$\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix} \begin{bmatrix} u_2^{k+1}(x_4) \\ u_2^{k+1}(x_5) \\ u_2^{k+1}(x_6) \\ u_2^{k+1}(x_7) \end{bmatrix} = \begin{bmatrix} f_4 + u_1^{k+1}(x_3) \\ f_5 \\ f_6 \\ f_7 + u_3^k(x_6) \end{bmatrix}$$

Example: 1D, 2 elements overlap (cont.)

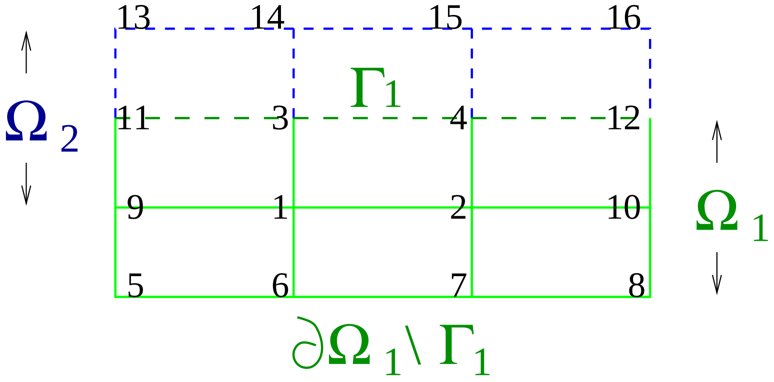
- (3) Solve for u_3^{k+1} in Ω_3:

$$\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix} \begin{bmatrix} u_3^{k+1}(x_7) \\ u_3^{k+1}(x_8) \\ u_3^{k+1}(x_9) \end{bmatrix} = \begin{bmatrix} f_7 + u_2^{k+1}(x_6) \\ f_8 \\ f_9 \end{bmatrix}$$

- $u^{k+1} = (u_1^{k+1}(x_1), u_1^{k+1}(x_2), u_1^{k+1}(x_3), u_2^{k+1}(x_4), u_2^{k+1}(x_5), u_2^{k+1}(x_6), u_3^{k+1}(x_7), u_3^{k+1}(x_8), u_3^{k+1}(x_9))$

- Related to but different from block GS. Will explain later.

Example: 2D



- Solve

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

- 1st half-step: (subscripts denote component number)

  Solve for $\begin{pmatrix} u_1^{k+1/2} \\ u_2^{k+1/2} \end{pmatrix}$, given boundary conditions:

$$\begin{cases} -\Delta u^{k+1/2} = f & \text{in } \Omega_1 \\ u^{k+1/2} = 0 & \text{on } \partial\Omega_1 \backslash \Gamma_1 \\ u_3^{k+1/2} = u_3^k, \quad u_4^{k+1/2} = u_4^k & \text{on } \Gamma_1 \end{cases}$$

  i.e.

$$\begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} u_1^{k+1/2} \\ u_2^{k+1/2} \end{pmatrix} = \begin{pmatrix} f_1 - u_3^k \\ f_2 - u_4^k \end{pmatrix}$$

Example: 2D (cont.)

- 2nd half-step:
  Solve for $\begin{pmatrix} u_3^{k+1} \\ u_4^{k+1} \end{pmatrix}$, given boundary conditions:

  $$\begin{cases} -\Delta u^{k+1} = f & \text{in } \Omega_2 \\ u^{k+1} = 0 & \text{on } \partial\Omega_2 \backslash \Gamma_2 \\ u_1^{k+1} = u_1^{k+1/2}, \ u_2^{k+1} = u_2^{k+1/2} & \text{on } \Gamma_2 \end{cases}$$

  Similarly, we have

  $$\begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} u_3^{k+1/2} \\ u_4^{k+1/2} \end{pmatrix} = \begin{pmatrix} f_3 - u_1^{k+1/2} \\ f_4 - u_2^{k+1/2} \end{pmatrix}$$

Matrix Interpretation

- Assume matching grid, i.e. the discretizations of the 2 subdomains coincide in the overlapping region.

Notation:

- Write the approx. solution u_i in Ω_i as:

$$
\begin{aligned}
u_i &= \begin{pmatrix} u_{\Omega_i} \\ u_{\partial\Omega_i\backslash\Gamma_i} \\ u_{\Gamma_i} \end{pmatrix} \\
&= \begin{pmatrix} u_i \text{ in the interior of } \Omega_i \\ u_i \text{ on the true boundary } \partial\Omega_i\backslash\Gamma_i \\ u_i \text{ on the artificial boundary } \Gamma_i \end{pmatrix}
\end{aligned}
$$

- Note: $u_{\partial\Omega_i\backslash\Gamma_i}$ are actually known; given by the Dirichlet boundary condition. They are kept as unknowns for convenience.

- Let A_i = discrete L on Ω_i. Similar to above, it can be written as:

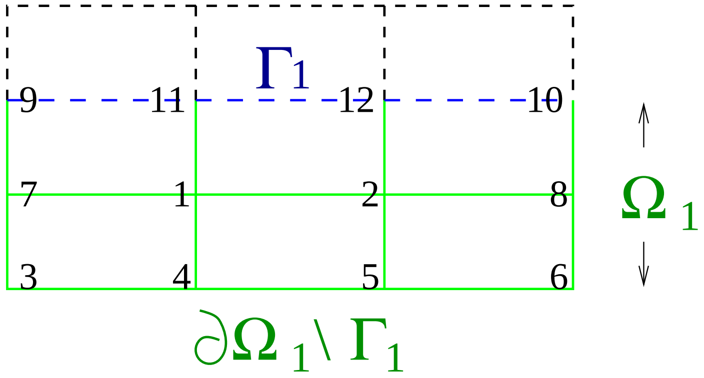
$A_i = (A_{\Omega_i}\ A_{\partial\Omega_i\backslash\Gamma_i}\ A_{\Gamma_i})$

A_{Ω_i} = coupling between interior nodes.

$A_{\partial\Omega_i\backslash\Gamma_i}$ = coupling between interior and true boundary nodes.

A_{Γ_i} = coupling between interior and artificial boundary nodes.

Example: $L = -\Delta$



- The components of u in Ω_1:

$$u_{\Omega_1} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad u_{\partial\Omega_1\setminus\Gamma_1} = \begin{pmatrix} u_3 \\ \vdots \\ u_{10} \end{pmatrix}, \quad u_{\Gamma_1} = \begin{pmatrix} u_{11} \\ u_{12} \end{pmatrix}$$

- The couplings of A_1 in Ω_1:

$$\begin{aligned} & A_1 u_1 \\ = \; & (A_{\Omega_1} \;\; A_{\partial\Omega_1\setminus\Gamma_i} \;\; A_{\Gamma_1}) \begin{pmatrix} u_{\Omega_1} \\ u_{\partial\Omega_1\setminus\Gamma_1} \\ u_{\Gamma_1} \end{pmatrix} \\ = \; & \begin{pmatrix} 4 & -1 & \vdots & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & \vdots & 0 & 0 \\ -1 & 4 & \vdots & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & \vdots & -1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \cdots \\ u_3 \\ \vdots \\ u_{10} \\ \cdots \\ u_{11} \\ u_{12} \end{pmatrix} \end{aligned}$$

- 1st row: difference eqn. at x_1.
  2nd row: difference eqn. at x_2.

Matrix Interpretation (cont.)

Using the new subscript notations, the alternating Schwarz algorithm can be written as the following 2 half-steps:

$$\left\{\begin{array}{rcl}
(A_{\Omega_1}\;\; A_{\partial\Omega_1\backslash\Gamma_1}\;\; A_{\Gamma_1})\begin{pmatrix} u_{\Omega_1}^{k+1} \\ u_{\partial\Omega_1\backslash\Gamma_1}^{k+1} \\ u_{\Gamma_1}^{k+1} \end{pmatrix} & = & f_1 \text{ in } \Omega_1 \\
u_{\partial\Omega_1\backslash\Gamma_1}^{k+1} & = & g_1 \text{ on } \partial\Omega_1\backslash\Gamma_1 \\
u_{\Gamma_1}^{k+1} & = & u_{\Omega_2}^{k}|_{\Gamma_1} \text{ on } \Gamma_1
\end{array}\right.$$

$$\left\{\begin{array}{rcl}
(A_{\Omega_2}\;\; A_{\partial\Omega_2\backslash\Gamma_2}\;\; A_{\Gamma_2})\begin{pmatrix} u_{\Omega_2}^{k+1} \\ u_{\partial\Omega_2\backslash\Gamma_2}^{k+1} \\ u_{\Gamma_2}^{k+1} \end{pmatrix} & = & f_1 \text{ in } \Omega_2 \\
u_{\partial\Omega_2\backslash\Gamma_2}^{k+1} & = & g_1 \text{ on } \partial\Omega_2\backslash\Gamma_2 \\
u_{\Gamma_2}^{k+1} & = & u_{\Omega_1}^{k+1}|_{\Gamma_2} \text{ on } \Gamma_2
\end{array}\right.$$

- Multiply A_i and u_i^{k+1} and move known quantities to the right-hand sides:

$$(*)\left\{\begin{array}{rcl}
A_{\Omega_1}u_{\Omega_1}^{k+1} & = & f_1 - A_{\partial\Omega_1\backslash\Gamma_1}g_1 - A_{\Gamma_1}u_{\Omega_2}^{k} \\
A_{\Omega_2}u_{\Omega_2}^{k+1} & = & f_2 - A_{\partial\Omega_2\backslash\Gamma_2}g_2 - A_{\Gamma_2}u_{\Omega_1}^{k+1}
\end{array}\right.$$

Matrix Interpretation (cont.)

Let $\tilde{f}_i = f_i - A_{\partial\Omega_i \backslash \Gamma_i} g_i$, which is the usual right-hand side for Ω_i after eliminating unknowns on the true boundary. Then (*) becomes:

$$(*)\begin{cases} A_{\Omega_1} u_{\Omega_1}^{k+1} = \tilde{f}_1 - A_{\Gamma_1} u_{\Omega_2}^{k} \\ A_{\Omega_2} u_{\Omega_2}^{k+1} = \tilde{f}_2 - A_{\Gamma_2} u_{\Omega_1}^{k+1} \end{cases}$$

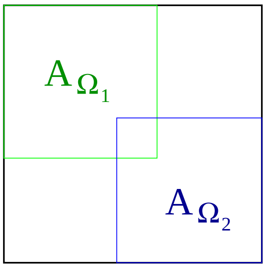
- This is precisely block GS method applied to the linear system:

$$\begin{pmatrix} A_{\Omega_1} & A_{\Gamma_1} \\ A_{\Omega_2} & A_{\Gamma_2} \end{pmatrix} \begin{pmatrix} u_{\Omega_1} \\ u_{\Omega_2} \end{pmatrix} = \begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{pmatrix}$$

- Note: The size of this linear system is bigger than n since the values of u on $\Omega_1 \cap \Omega_2 \neq \emptyset$ are duplicated in u_{Ω_1} and u_{Ω_2}.

- Partitioning of matrix A_Ω:

$A_\Omega =$



Matrix Operator Form

- Write the alternating Schwarz as 2 half-steps:

$$\left\{\begin{array}{rcl} u^{k+1/2} & \leftarrow & u^k + \begin{pmatrix} A_{\Omega_1}^{-1} & 0 \\ 0 & 0 \end{pmatrix} (f - Au^k) \\ u^{k+1} & \leftarrow & u^{k+1/2} + \begin{pmatrix} 0 & 0 \\ 0 & A_{\Omega_2}^{-1} \end{pmatrix} (f - Au^{k+1/2}) \end{array}\right.$$

- Define the restriction operator R_i (rectangular matrix) as:

$$\begin{array}{rcl} u_{\Omega_1} & = & R_1 u \equiv (I\ 0) \begin{pmatrix} u_{\Omega_1} \\ u_{\Omega \backslash \Omega_1} \end{pmatrix} \\ u_{\Omega_2} & = & R_2 u \equiv (0\ I) \begin{pmatrix} u_{\Omega \backslash \Omega_2} \\ u_{\Omega_2} \end{pmatrix} \end{array}$$

  i.e. $R_i u$ restricts u to the subdomain Ω_i.

- Note: $A_{\Omega_i} = R_i A R_i^T$. Hence, alternating Schwarz can be written as:

$$\left\{\begin{array}{rcl} u^{k+1/2} & \leftarrow & u^k + R_1^T (R_1 A R_1^T)^{-1} R_1 (f - Au^k) \\ u^{k+1} & \leftarrow & u^{k+1/2} + R_2^T (R_2 A R_2^T)^{-1} R_2 (f - Au^{k+1/2}) \end{array}\right.$$

- No need to form R_i in practice; only used in analysis.

- Define $B_i = R_i^T(R_iAR_i^T)^{-1}R_i$, i.e. B_i restricts the residual to subdomain Ω_i, solves the subdomain problem to generate a correction, and extends back onto the entire domain.
  - ▷ Just like R_i, never form B_i in practice.

- Then the alternating Schwarz iteration can be written as:

$$u^{k+1} \leftarrow u^k + (B_1 + B_2 - B_2AB_1)(f - Au^k)$$

  i.e. The preconditioner B_{MS} ($\approx A^{-1}$) given by the Schwarz method is:

$$B_{MS} = B_1 + B_2 - B_2AB_1$$

- Define the error of u^k by: $e^k = u - u^k$. By direct calculation,

$$\begin{aligned} e^{k+1} &\leftarrow e^k - (B_1 + B_2 - B_2AB_1)Ae^k \\ &= [I - (B_1 + B_2 - B_2AB_1)A]e^k \\ &= (I - B_2A)(I - B_1A)e^k \end{aligned}$$

  Hence, the iteration matrix is a product of 2 matrices → multiplicative Schwarz.

Symmetrized Multiplicative Schwarz

- The preconditioner B_{MS} is *not symmetric*, and hence cannot be used as a preconditioner for CG, even if A is SPD.

- Can symmetrize the multiplicative Schwarz by adding one more intermediate step:

$$\begin{aligned} u^{k+1/3} &\leftarrow u^k + B_1(f - Au^k) \\ u^{k+2/3} &\leftarrow u^{k+1/3} + B_2(f - Au^{k+1/3}) \\ u^{k+1} &\leftarrow u^{k+2/3} + B_1(f - Au^{k+2/3}) \end{aligned}$$

  i.e. Solving the subdomain Ω_1 problem again.

- The preconditioner B_{SMS} of the symmetric multiplicative Schwarz is:

$$B_{SMS} = B_1 + (I - B_1 A) B_2 (I - AB_1)$$

Additive Schwarz Method

- If we do **not** use the most updated values when solving Ω_2, the Schwarz iteration becomes:

$$\begin{cases} u^{k+1/2} & \leftarrow & u^k + \begin{pmatrix} A_{\Omega_1}^{-1} & 0 \\ 0 & 0 \end{pmatrix} (f - Au^k) \\ u^{k+1} & \leftarrow & u^{k+1/2} + \begin{pmatrix} 0 & 0 \\ 0 & A_{\Omega_2}^{-1} \end{pmatrix} (f - A{\color{red}u^k}) \end{cases}$$

- Both updates can be performed simultaneously → parallelism.

- Combing the two half-steps:

$$u^{k+1} \leftarrow u^k + \left(\begin{pmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & A_2^{-1} \end{pmatrix} \right) (f - Au^k)$$

  i.e.

$$u^{k+1} \leftarrow u^k + (B_1 + B_2)(f - Au^k)$$

- The preconditioner B_{AS} given by the additive Schwarz is:

$$B_{AS} = B_1 + B_2$$

- Can be viewed as generalized block Jacobi method.

Additive vs Multiplicative

- Analogous to Jacobi vs Gauss-Seidel
- Convergence rate of additive Schwarz is slower than multiplicative Schwarz. Typically, MS requires half as many iterations as AS.
- AS is easily parallelizable. Each subdomain problem can be solved independently in parallel whereas MS solves the subdomain problem sequentially.
- Parallelize MS using multicoloring.

Numerical Example

PDE:

$$\begin{aligned} -\Delta u &= xe^y \quad \text{in } \Omega \\ u &= -xe^y \quad \text{on } \partial\Omega \end{aligned}$$

Domain and domain partition: same as previous example.

Convergence results: iteration counts

n	Overlap size	MS	AS
11	1	2	4
11	2	2	3
11	3	2	3
21	1	3	5
21	2	2	4
21	3	2	4
31	1	3	5
31	2	3	5
31	3	2	4

- Both AS and MS are insentive to overlap size.
- About a factor of 2 difference between the number of iterations for MS and AS methods.

Practical Considerations

- For relatively large subdomain cases, exact subdomain solves may be too expensive to carry out.
- Since the Schwarz method is typically used as preconditioner for a Krylov subsp. method, we may solve the subdomain problems approximately, i.e. $A_{\Omega_i}^{-1} \approx \tilde{A}_{\Omega_i}^{-1}$.
- Krylov subsp theory requires that $\tilde{A}_{\Omega_1}^{-1}$ is a linear operator, e.g. a fixed number of sweeps of GS, or PCG solved to machine precision.
- Several steps of a Krylov subsp method is **not** a linear operator.
- However, experience indicates that if the local problems are solved accurately enough, convergence of the outer iteration is not affected much even if the local solver is a nonlinear operator such as a Krylov subsp method.