# Introduction to Robotics (CS225A) Handout #05

(Spring 2007-08)

## Solution Set #1

1. Solution by Khatib's method (Explicit Form), with $q_1 = \theta_1$, $q_2 = \theta_2$, and $q_3 = \theta_3$.
   The equations of motion for this manipulator is,

$$M(\Theta)\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\\ \ddot{\theta}_3\end{bmatrix} + B(\Theta)\begin{bmatrix}\dot{\theta}_1\dot{\theta}_2\\ \dot{\theta}_1\dot{\theta}_3\\ \dot{\theta}_2\dot{\theta}_3\end{bmatrix} + C(\Theta)\begin{bmatrix}\dot{\theta}_1^2\\ \dot{\theta}_2^2\\ \dot{\theta}_3^2\end{bmatrix} + G(\Theta) = \begin{bmatrix}\tau_1\\ \tau_2\\ \tau_3\end{bmatrix}$$

   (a) Forward Kinematics

$${}^0_E T = {}^0_1T\,{}^1_2T\,{}^2_3T\,{}^3_E T = \begin{bmatrix} c_{123} & -s_{123} & 0 & L_1c_1 + L_2c_{12} + L_3c_{123}\\ s_{123} & c_{123} & 0 & L_1s_1 + L_2s_{12} + L_3s_{123}\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

   (b) Jacobian matrix

$$J(\Theta) = \begin{bmatrix} -L_1s_1 - L_2s_{12} - L_3s_{123} & -L_2s_{12} - L_3s_{123} & -L_3s_{123}\\ L_1c_1 + L_2c_{12} + L_3c_{123} & L_2c_{12} + L_3c_{123} & L_3c_{123}\\ 1 & 1 & 1\end{bmatrix}$$

   (c) Kinetic energy matrix

$$M(\Theta) = m_1J_{v_1}^T J_{v_1} + J_{\omega_1}^{T\ C_1}I_1J_{\omega_1} + m_2J_{v_2}^T J_{v_2} + J_{\omega_2}^{T\ C_2}I_2J_{\omega_2} + m_3J_{v_3}^T J_{v_3} + J_{\omega_3}^{T\ C_3}I_3J_{\omega_3}$$

$${}^0P_{C1} = \begin{bmatrix} r_1c_1\\ r_1s_1\\ 0\end{bmatrix},\ {}^0P_{C2} = \begin{bmatrix} L_1c_1 + r_2c_{12}\\ L_1s_1 + r_2s_{12}\\ 0\end{bmatrix},\ {}^0P_{C3} = \begin{bmatrix} L_1c_1 + L_2c_{12} + r_3c_{123}\\ L_1s_1 + L_2s_{12} + r_3s_{123}\\ 0\end{bmatrix}$$

$$J_{v_1} = \begin{bmatrix} -r_1s_1 & 0 & 0\\ r_1c_1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\ J_{v_2} = \begin{bmatrix} -L_1s_1 - r_2s_{12} & -r_2s_{12} & 0\\ L_1c_1 + r_2c_{12} & r_2c_{12} & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$J_{v_3} = \begin{bmatrix} -L_1s_1 - L_2s_{12} - r_3s_{123} & -L_2s_{12} - r_3s_{123} & -r_3s_{123}\\ L_1c_1 + L_2c_{12} + r_3c_{123} & L_2c_{12} + r_3c_{123} & r_3c_{123}\\ 0 & 0 & 0\end{bmatrix}$$

$${}^{C_i}I_i = \begin{bmatrix} {}^{C_i}I_{xx} & 0 & 0\\ 0 & {}^{C_i}I_{yy} & 0\\ 0 & 0 & {}^{C_i}I_{zz}\end{bmatrix} \text{ for } i = 1, 2, 3.$$

$$J_{\omega_1} = \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0\end{bmatrix},\ J_{\omega_2} = \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 1 & 0\end{bmatrix},\ J_{\omega_3} = \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 1 & 1\end{bmatrix}$$

$$M(\Theta) = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

$$\begin{aligned}
m_{11} &= m_1 r_1^2 + {}^{C_1}I_{zz_1} + m_2(r_2^2 + L_1^2 + 2L_1 r_2 c_2) + {}^{C_2}I_{zz_2} \\
&\quad + m_3(r_3^2 + L_1^2 + L_2^2 + 2(L_1 L_2 c_2 + L_1 r_3 c_{23} + L_2 r_3 c_3)) + {}^{C_3}I_{zz_3} \\
m_{12} &= m_2(r_2^2 + L_1 r_2 c_2) + {}^{C_2}I_{zz_2} + m_3(r_3^2 + L_2^2 + L_1 L_2 c_2 + L_1 r_3 c_{23} + 2L_2 r_3 c_3) + {}^{C_3}I_{zz_3} \\
m_{13} &= m_3(r_3^2 + L_1 r_3 c_{23} + L_2 r_3 c_3) + {}^{C_3}I_{zz_3} \\
m_{21} &= m_{12} \\
m_{22} &= m_2 r_2^2 + {}^{C_2}I_{zz_2} + m_3(r_3^2 + L_2^2 + 2L_2 r_3 c_3) + {}^{C_3}I_{zz_3} \\
m_{23} &= m_3(r_3^2 + L_2 r_3 c_3) + {}^{C_3}I_{zz_3} \\
m_{31} &= m_{13} \\
m_{32} &= m_{23} \\
m_{33} &= m_3 r_3^2 + {}^{C_3}I_{zz_3}
\end{aligned}$$

(d) Coriolis matrix

$$B(\Theta) = \begin{bmatrix} 2b_{112} & 2b_{113} & 2b_{123} \\ 2b_{212} & 2b_{213} & 2b_{223} \\ 2b_{312} & 2b_{313} & 2b_{323} \end{bmatrix}$$

$$= \begin{bmatrix} -2(m_2 L_1 r_2 s_2 + m_3 L_1(L_2 s_2 + r_3 s_{23})) & -2m_3(L_1 r_3 s_{23} + L_2 r_3 s_3) & -2m_3(L_1 r_3 s_{23} + L_2 r_3 s_3) \\ 0 & -2m_3 L_2 r_3 s_3 & -2m_3 L_2 r_3 s_3 \\ 2m_3 L_2 r_3 s_3 & 0 & 0 \end{bmatrix}$$

(e) Centrifugal matrix

$$C(\Theta) = \begin{bmatrix} b_{111} & b_{122} & b_{133} \\ b_{211} & b_{222} & b_{233} \\ b_{311} & b_{322} & b_{333} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -m_2 L_1 r_2 s_2 - m_3 L_1(L_2 s_2 + r_3 s_{23}) & -m_3(L_1 r_3 s_{23} + L_2 r_3 s_3) \\ m_2 L_1 r_2 s_2 + m_3 L_1(L_2 s_2 + r_3 s_{23}) & 0 & -m_3 L_2 r_3 s_3 \\ m_3(L_1 r_3 s_{23} + L_2 r_3 s_3) & m_3 L_2 r_3 s_3 & 0 \end{bmatrix}$$

(f) Gravity matrix with $\mathbf{g} = [0\ g\ 0]^T$.

$$G(\Theta) = -(m_1 J_{v_1}^T \mathbf{g} + m_2 J_{v_2}^T \mathbf{g} + m_3 J_{v_3}^T \mathbf{g}) = \begin{bmatrix} -(m_1 r_1 c_1 + m_2(L_1 c_1 + r_2 c_{12}) + m_3(L_1 c_1 + L_2 c_{12} + r_3 c_{123}))g \\ -(m_2 r_2 c_{12} + m_3(L_2 c_{12} + r_3 c_{123}))g \\ -m_3 r_3 c_{123} g \end{bmatrix}$$